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Approximation of Surface Measures in a Locally Convex Space

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Abstract—The main result of the paper is an analog of the surface layer theorem for measures given on a locally convex space with a continuously and densely embedded Hilbert subspace (for a surface of finite codimension). Earlier, the surface layer theorem was proved only for Banach spaces: for surfaces of codimension 1 by Uglanov (1979) and for surfaces of an arbitrary finite codimension by Yakhlakov (1990). In these works, the definition of the surface layer and the proof of the theorem essentially use the fact that the original space is equipped with a norm.

Key words: *surface layer theorem, locally convex space, continuously and densely embedded Hilbert subspace, Fréchet derivative.*

The main result of this paper is to obtain an analog of the surface layer theorem for a measure given on a locally convex space with continuously and densely embedded Hilbert subspace (for a surface of finite codimension). Earlier, the surface layer theorem was proved only for Banach spaces: in [1] and [2] for surfaces of codimension 1 and in [3] for surfaces of a finite codimension. In these works, the definition of the surface layer and the proof of the theorem essentially use the fact that the original space is equipped with a norm.

In [4] the approach related to the approximation of a surface (Wiener) measure by normed measures of neighborhoods of the surface (the surface layers) was successfully used for surfaces of infinite codimension.

1. DEFINITIONS AND AUXILIARY RESULTS

In what follows, all vector spaces are considered over the field of real numbers. All topological spaces are Hausdorff spaces. Throughout the paper, E is a locally convex space, E' is its dual, H is a vector subspace of the space E endowed with Hilbert space structure with respect to the inner product $(\,\cdot\,,\,\cdot\,)$ (the corresponding norm is denoted by $\|\cdot\|$), and H' is its dual. In this case H is dense in E, the identical embedding of H in E is continuous, and $\mathcal{L}(\,\cdot\,,\,\cdot\,)$ is the space of continuous linear operators.

Since H is dense in E, the mapping $E' \to H'$ determined by the restriction of continuous linear functionals on E to the subspace H (adjoint to the embedding of H in E) is injective. We identify H' with H by the Riesz theorem and denote the corresponding mapping $E' \to H$ by j_H.

By R_a we denote the finite-dimensional subspace generated by a system $a = \{a_1, \dots, a_n\}$ of linearly independent vectors from H. We show that there is an isometry between R_a and $\mathbb{R}^n$ if we assume that

$$t_i \frac{a_i}{\|a_i\|} \leftrightarrow t_i, \qquad t_i \in \mathbb{R}.$$

In this sense (if the system of vectors a is fixed), we identify R_a with $\mathbb{R}^n$.

Throughout the paper, the differentiability of a function means its differentiability in the sense of Fréchet [5].

Definition 1. Suppose that $D_F \subset E$ is a domain and $F = (F_1, \dots, F_n)\colon D_F \to \mathbb{R}^n$ is a continuous function with the derivative $F'\colon D_F \to H'^n \cong H^n$ along the subspace H. Suppose also that

$$S \stackrel{\text{def}}{=} \{x \in D_F : F(x) = 0\}$$

and, at each point $x \in S$, the system of vectors $\{F_1'(x), \dots, F_n'(x)\}$ is linearly independent in H. The n-dimensional space spanned by these vectors is called the *subspace normal* to S at the point x.

Suppose that the set S satisfies the assumptions of Definition 1, $n^S(x)$ is the system of orthonormal vectors $\{n_1^S(x), \dots, n_n^S(x)\}$ obtained from the system $\{F_1'(x), \dots, F_n'(x)\}$ by Schmidt orthogonalization. Suppose also that $B_x(\varepsilon) = \{x \in R_{n^S(x)} : \|x\| < \varepsilon\}$ is a ball in the normal space $R_{n^S(x)}$. For an arbitrary subset $B \subset S$ and each $\varepsilon > 0$, we define the set

$$B^\varepsilon = \{x + v : x \in B,\ v \in B_x(\varepsilon)\}.$$

Definition 2. A set $G \subset E$ is called a *surface*[1] if it satisfies Conditions 1–3 given below.

Condition 1. The set G is the graph of a continuous function f defined on a convex domain $\mathcal{U}$ of a closed subspace T_b of codimension n of the space E and ranging in the space R_b, where $b = \{b_1, \dots, b_n\}$ is the system of orthonormal vectors in H orthogonal in H to the subspace $H_b \stackrel{\text{def}}{=} T_b \cap H$.

By the symbol P_b we denote the projection operator on the subspace T_b along the subspace R_b.

Remark 1. The subspace H_b is everywhere dense in the space T_b and is a closed subspace of codimension n of the space H.

Remark 2. Because of the isomorphism

$$T_b \times \mathbb{R}^n \to E, \qquad \bigl(x, (t_1, \dots, t_n)\bigr) \mapsto x + \sum_{i=1}^n t_i f_i(x),$$

in what follows (where it is convenient and cannot lead to a misunderstanding), we identify the spaces E and $T_b \times \mathbb{R}^n$; in this case we assume that the function f ranges in $\mathbb{R}^n$ and sometimes write f in the form $(f_1, \dots, f_n)$, where $f_i\colon \mathcal{U} \to \mathbb{R}$. Similarly, we identify the functions defined on E and the corresponding functions defined on $T_b \times \mathbb{R}^n$ (in a natural way, using the above isomorphism).

Condition 2. The function f has continuous

first-order derivative $f'\colon \mathcal{U} \to H_b'^n \cong H_b^n$;
second-order derivative $f''\colon \mathcal{U} \to \mathcal{L}(H_b, H_b')^n \cong \mathcal{L}(H_b, H_b)^n$;
third-order derivative $f'''\colon \mathcal{U} \to \mathcal{L}(H_b, \mathcal{L}(H_b, H_b))^n \cong \mathcal{L}(H_b, \mathcal{L}(H_b, H_b))^n$

along the subspace H, and there exists a constant $K_f > 0$ such that for all $x \in \mathcal{U}$ and $0 \le i \le n$ the following estimates hold:

$$\|f_i'(x)\| \le K_f, \qquad \|f_i''(x)\|_{\mathcal{L}(H_b, H_b)} \le K_f, \qquad \|f_i'''(x)\|_{\mathcal{L}(H_b, \mathcal{L}(H_b, H_b))} \le K_f,$$

where $\|\cdot\|_{\mathcal{L}(H_b, H_b)}$ and $\|\cdot\|_{\mathcal{L}(H_b, \mathcal{L}(H_b, H_b))}$ are operator norms on the corresponding spaces.

[1]This definition of surface does not coincide with the definition given in [1, 2, 6], where the notion of surface is understood as a more general object. Hence we can apply the results of [1, 2, 6] to our surfaces (and, in what follows, we shall do this without giving any references).

Remark 3. The set G can be represented in the form

$$G = \{(x, v) \in \mathcal{U} \times \mathbb{R}^n : \Phi(x, v) = 0\},$$

where the function Φ is defined as follows:

$$\Phi : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad (x, v) \mapsto v - f(x).$$

The function Φ is differentiable along the subspace H (because the function f is differentiable along H_b), and the system of vectors

$$\Phi'(x, v) = \{\Phi_1'(x, t_1), \dots, \Phi_n'(x, t_n)\},$$

where $\Phi_i' : \mathcal{U} \times \mathbb{R} \to H' \cong H$, $v = (t_1, \dots, t_n)$, is linearly independent at each point $(x, v) \in \mathcal{U} \times \mathbb{R}^n$. Indeed, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in H, and let $e_i = b_i$ for all $0 \le i \le n$. Then

$$\Phi_i'(x, t_i) = \sum_{k=1}^{\infty} (\Phi'(x, t_i), e_k) e_k = b_i - \sum_{k=2}^{\infty} (f_i'(x), e_k) e_k = b_i - f_i'(x)$$

for all $0 \le i \le n$. Since for all $x \in \mathcal{U}$, the vectors $f_i'(x)$ belong to the space H_b which is orthogonal to R_b, the system of vectors $\Phi'(x, v)$ is linearly independent because of the orthogonality of the system of vectors b. Hence the system of vectors $n^G(x, v)$ is defined at each point $(x, v) \in G$. We show that for the system $n^G(x, v)$ the set $\det\{(b_i, n_j^G(x, v))\}$ is bounded away from zero by some constant $\sigma > 0$. Indeed, let $\{N_1(x, v), \dots, N_n(x, v)\}$ be the system of vectors obtained from the system $\Phi'(x, v)$ by Schmidt orthogonalization. It is easy to see that

$$\det\{(b_i, n_j^G(x, v))\} = \left(\prod_{i=1}^{n} \|N_i(x, v)\| \right)^{-1}.$$

But, by Condition 2, the values of $\|N_i(x, v)\|$ are bounded on $\mathcal{U} \times \mathbb{R}^n$.

Remark 4. Since there exists a bijection

$$g_f : \mathcal{U} \to G, \qquad x \mapsto x + \sum_{i=1}^{n} f_i(x) b_i$$

of the domain $\mathcal{U}$ onto the set G, in what follows, we assume that the domain $\mathcal{U}$ is the domain of definition of the mapping n^G and, instead of $n^G(x, v)$, where $(x, v) \in G$, simply write $n^G(x)$, where we assumed that $x \in \mathcal{U}$.

Condition 3. Let $v = (t_1, \dots, t_n) \in \mathbb{R}^n$, and let $B(\varepsilon) = \{v \in \mathbb{R}^n : \|v\| < \varepsilon\}$. There exists an $\varepsilon_b > 0$ such that the mapping g_f can be extended to a one-to-one mapping (onto the image)

$$\varphi : \mathcal{U} \times B(\varepsilon_b) \to E, \qquad (x, v) \mapsto g_f(x) + \sum_{i=1}^{n} t_i n_i^G(x), \qquad G^{\varepsilon_b} \equiv \varphi(\mathcal{U} \times B(\varepsilon_b)) \subset E,$$

such that the inverse mapping $\psi = \varphi^{-1}$ is continuous on G^{ε_b}.

In what follows, for simplicity, we perform all the proofs for surfaces of codimension 1, although the statements themselves hold for surfaces of any arbitrary finite codimension.

Lemma 1. *For all* $x \in \mathcal{U}$, *we have* $(n^G)'(x) \in \mathcal{L}(H_b, H)^n$ *and* $(n^G)''(x) \in \mathcal{L}(H_b, \mathcal{L}(H_b, H))^n$, *and there exists a constant* $K_n > 0$ *such that the estimates*

$$\|(n_i^G)'(x)\|_{\mathcal{L}(H_b,H)} < K_n, \qquad \|(n_i^G)''(x)\|_{\mathcal{L}(H_b,\mathcal{L}(H_b,H))} < K_n \tag{1}$$

hold for all $0 \le i \le n$.

Proof. Differentiating the relation

$$n^G(x) = \frac{b - f'(x)}{\sqrt{1 + \|f'(x)\|^2}}$$

along an arbitrary direction $h \in H_b$, we obtain

$$(n^G)'(x)h = \frac{(f'(x) - b)(f'(x), f''(x)h)}{(1 + \|f'(x)\|^2)^{3/2}} - \frac{f''(x)h}{(1 + \|f'(x)\|^2)^{1/2}}. \tag{2}$$

This and Condition 2 imply the inclusion $(n^G)'(x) \in \mathcal{L}(H_b, H)$, and relation (2) implies the estimate

$$\|(n^G)'(x)\| \le \big(\|f'(x)\| + 1\big)\|f'(x)\| \cdot \|f''(x)\| + \|f''(x)\| < (K_f + 1)K_f^2 + K_f.$$

Differentiating (2) along an arbitrary vector $h_1 \in H_b$ and again taking into account Condition 2, we obtain the second estimate in (1). $\square$

Lemma 2. *Suppose that* X *and* X_1 *are locally convex spaces,* $X_1 \subset X$ *is continuously embedded, and the mapping* $\Psi\colon X \supset D(\Psi) \to X$ *has the derivative* $\Psi'\colon D(\Psi) \to \mathcal{L}(X_1, X_1)$ (*with respect to* X_1). *Suppose that for the curve* $\gamma\colon [0,1] \to D(\Psi)$ *the derivative* $(d\gamma/dt)(t_0)$ *in the topology of the space* X_1 *exists at some point* $t_0 \in [0,1]$. *Then at the point* t_0 *the composite function* $\Psi \circ \gamma$ *has a derivative, and*

$$\frac{d}{dt}(\Psi \circ \gamma)(t)\bigg|_{t=t_0} = \Psi'(\gamma(t_0))\,\frac{d\gamma}{dt}(t_0).$$

Remark 5. The fact that the derivative $(d\gamma/dt)(t_0)$ exists in the topology of the space X_1 means that, for all t in some neighborhood of the point t_0, we have $\gamma(t) - \gamma(t_0) \in X_1$ and the limit

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \equiv \frac{d\gamma}{dt}(t_0)$$

exists in the topology of the space X_1.

Proof. The proof of Lemma 2 is obvious. $\square$

Lemma 3. *The mapping* φ *is continuous on* $\mathcal{U} \times B(\varepsilon_b)$ *and possesses the first-order derivative* $\varphi'\colon \mathcal{U} \times B(\varepsilon_b) \to \mathcal{L}(H, E)$ *and the second-order derivative* $\varphi''\colon \mathcal{U} \times B(\varepsilon_b) \to \mathcal{L}(H, \mathcal{L}(H, E))$ *along the subspace* H; *moreover,* φ' *ranges in the space* $\mathcal{L}(H, H)$, φ'' *ranges in the space* $\mathcal{L}(H, \mathcal{L}(H, H))$, *and there exists a constant* $K_\varphi > 0$ *such that*

$$\|\varphi'(x)\|_{\mathcal{L}(H,H)} < K_\varphi, \qquad \|\varphi''(x)\|_{\mathcal{L}(H,\mathcal{L}(H,H))} < K_\varphi$$

for all $x \in \mathcal{U} \times B(\varepsilon_b)$.

Proof. The mapping φ is continuous by its definition (Condition 3) and by the fact the mapping n^G is continuous. Let us verify the differentiability. If $(x,t)\in\mathcal{U}\times(-\varepsilon_b,\varepsilon_b)$, $h\in H$, and $\tau\in\mathbb{R}$, then

$$\begin{aligned}\varphi(x+tb+\tau h)&=\varphi\big(x+\tau P_b h+(t+\tau(h,b))b\big)\\&=x+\tau P_b h+f(x+\tau P_b h)b+(t+\tau(b,h))n^G(x+\tau P_b h).\end{aligned}$$

Calculating the derivative of this expression with respect to τ at the point $\tau=0$ and using Lemmas 2 and 1, we obtain

$$\begin{aligned}\varphi'(x,t)h=\frac{d\varphi}{d\tau}\bigg|_{\tau=0}&=P_b h+(f'(x),P_b h)b+(b,h)n^G(x)+t(n^G)'(x)P_b h\\&=(f'(x)-b,h)b+(b,h)n^G(x)+h+t(n^G)'(x)P_b h.\end{aligned}\tag{3}$$

By Lemma 1, we have $(n^G)'(x)\in\mathcal{L}(H_b,H)$ and hence $\varphi'(x,t)\in\mathcal{L}(H,H)$. Differentiating relation (3) and again taking Lemma 1 into account, we obtain $\varphi''(x,t)\in\mathcal{L}(H,\mathcal{L}(H,H))$. The boundedness of the derivatives φ' and φ'' follows from Condition 2 and Lemma 1. The proof of the lemma is complete. $\square$

Lemma 4. *There exists an $\varepsilon_b'<\varepsilon_b$ such that the mapping ψ has the first-order derivative $\psi'\colon G^{\varepsilon_b'}\to\mathcal{L}(H,H)$ and the second-order derivative $\psi''\colon G^{\varepsilon_b'}\to\mathcal{L}(H,\mathcal{L}(H,H))$ along the subspace H, and there exists a constant K_ψ such that, for all $x\in G^{\varepsilon_b'}$,*

$$\|\psi'(x)\|_{\mathcal{L}(H,H)}<K_\psi,\qquad \|\psi''(x)\|_{\mathcal{L}(H,\mathcal{L}(H,H))}<K_\psi.$$

In what follows, we omit the subscripts on the norms $\|\cdot\|_{\mathcal{L}(H,H)}$ and $\|\cdot\|_{\mathcal{L}(H,\mathcal{L}(H,H))}$.

Proof. For $\omega\in\mathcal{U}\times(-\varepsilon_b,\varepsilon_b)$, we define the function

$$\varphi_\omega\colon(\mathcal{U}\times(-\varepsilon_b,\varepsilon_b)-\omega)\cap H\to H,\qquad u\mapsto\varphi(\omega+u)-\varphi(\omega).$$

Similarly, for each $\omega\in G^{\varepsilon_b}$, we define the function

$$\psi_\omega\colon G^{\varepsilon_b}\cap H\to H,\qquad u\mapsto\psi(\omega+u)-\psi(\omega).$$

We choose an arbitrary point $\omega_0\in G^{\varepsilon_b}$. A straightforward verification shows that

$$\varphi_{\psi(\omega_0)}(\psi_{\omega_0}(u))=u.$$

We denote $\psi(\omega_0)$ by the symbol w_0 and consider the equation $\varphi_{w_0}(v)=u$. Next, we show that, for some $\varepsilon_b'<\varepsilon_b$, the point $\omega_0\in G^{\varepsilon_b'}$ satisfies all the assumptions of the theorem stating that the inverse function ψ_{ω_0} is differentiable at the point $u=0$ [7]. Indeed, for all $w\in\mathcal{U}\times(-\varepsilon_b,\varepsilon_b)$, we have $(\varphi_w)'(v)=\varphi'(w+v)\in\mathcal{L}(H,H)$, and $(\varphi_w)'$ is a continuous function on its domain of definition, since it is differentiable on this domain. Then we find an $\varepsilon_b'<\varepsilon_b$ such that for $w\in\mathcal{U}\times(-\varepsilon_b',\varepsilon_b')$ the operator $(\varphi_w)'(0)$ is invertible. First, we assume that $w\in\mathcal{U}$. We write w as $w=(x,0)$, where $x\in\mathcal{U}$. By Lemma 3, we have $(\varphi_w)'=\varphi'(x,0)\in\mathcal{L}(H,H)$. Let us show that $\varphi'(x,0)$ is a bijection, which, by the Banach theorem on the inverse operator, implies the existence of $(\varphi'(x,0))^{-1}\in\mathcal{L}(H,H)$. For $h\in H$, it follows from the proof of Lemma 3 that

$$\varphi'(x,0)h=(f'(x)-b,h)b+(b,h)n^G(x)+h=(K+I)h,$$

where $K=(f'(x)-b,h)b+(b,h)n^G(x)$ is a finite-dimensional operator such that $\ker(K+I)=\{0\}$. Indeed, each vector $h\in\ker(K+I)$ can be written as $h=-Kh$, and hence $h\in\operatorname{Im}K$. Therefore, in the case in which the vectors b and $n^G(x)$ are linearly independent, there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $h = \alpha b + \beta n^G(x)$. Substituting this expression in the equation

$$(K + I)h = 0 \tag{4}$$

and taking into account the fact that

$$n^G(x) = \frac{b - f'(x)}{\sqrt{1 + \|f'(x)\|^2}},$$

we obtain the following two equations for the numbers α and β:

$$-\beta\sqrt{1 + \|f'(x)\|^2} = 0, \qquad \alpha + \beta\big(1 + (b, n^G(x))\big) = 0.$$

Obviously, this system has only a trivial solution, and hence $h = 0$. If the vectors b and $n^G(x)$ are linearly dependent, then it is easy to see that $b = n^G(x)$. In this case the solution of Eq. (4) has the form $h = \alpha b$, where $\alpha \in \mathbb{R}$. Substituting this expression in (4), we again obtain $h = 0$. Hence, by the Fredholm theorem, the operator $K + I$ $(= \varphi'(x, 0))$ is invertible. Thus, for all $w \in \mathcal{U}$, the operator $(\varphi_w)'(0) = \varphi'(w)$ is invertible and, by the theorem on the differentiability of the inverse function [7], for all $\omega \in G$, we have

$$\psi'(\omega) = (\psi_\omega)'(0) = \big((\varphi_w)'(0)\big)^{-1} \in \mathcal{L}(H, H), \qquad w = \psi(\omega).$$

Further, let us show that $\psi'(\omega)$ (as a function of ω) is bounded on G with respect to the norm of the space $\mathcal{L}(H, H)$. For $\omega \in G$, we shall find the explicit form of the operator $\psi'(\omega)$. To this end, we set $\omega = (x, f(x))$ and, for all $h' \in H$, we solve the equation

$$(f'(x) - b, h)b + (b, h)n^G(x) + h = h' \tag{5}$$

with respect to h. First, we consider the case where $b \neq n^G(x)$ (which means that the vectors b and $n^G(x)$ are linearly independent). Taking into account the relation

$$h = h' - (f'(x) - b, h)b - (b, h)n^G(x),$$

we seek h in the form $h = h' + \alpha b + \beta n^G(x)$. Substituting this in Eq. (5), we obtain the following expressions for α and β:

$$\beta = \frac{(f'(x) - b, h')}{\sqrt{1 + \|f'(x)\|^2}}, \qquad \alpha = -\beta - (b, h' + \beta n^G(x)).$$

Using these relations and the fact that the solution is unique, we obtain the following estimate for $\psi'(\omega)$:

$$\|\psi'(\omega)h'\| = \|h' + \alpha b + \beta n^G(x)\| \le \|h'\| + |\alpha| + |\beta|.$$

Then we can write

$$|\beta| \le \frac{\|f'(x) - b\|}{\sqrt{1 + \|f'(x)\|^2}}\|h'\| = \|h'\|, \qquad |\alpha| \le 2|\beta| + \|h'\| \le 3\|h'\|.$$

Hence $\|\psi'(\omega)\| \le 5$ for all $\omega \in G$. Now we assume that $b = n^G(x)$. We seek the solution of Eq. (5) in the form $h = h' + \alpha b$. Substituting this expression in Eq. (5), we obtain $\alpha = -(f'(x), h')$. Hence $\|\psi'(\omega)\| \le 1 + K_f$.

Next, we rewrite the expression for $\varphi'(x,t)$, where $x \in \mathcal{U}$ and $t \in (-\varepsilon_b, \varepsilon_b)$ (see the proof of Lemma 3) as follows:

$$\varphi'(x,t)h = (f'(x) - b, h)b + (b, h)n^G(x) + h + t(n^G)'(x)P_b h = A_0 h + \Delta A_t h,$$

where $A_0 = \varphi'(x,0)$ and $\Delta A_t = t(n^G)'(x)P_b$. Lemma 1 implies the estimate

$$\|\Delta A_t\| \le \|(n^G)'(x)\| t \le K_n t;$$

hence there exists a $t_0 > 0$ such that, for all $t \in (-t_0, t_0)$,

$$\|\Delta A_t\| < \frac{1}{2K}, \tag{6}$$

where $K = \max\{5, K_f + 1\}$. Hence we have $\|\Delta A_t\| < \|A_0^{-1}\|^{-1}$. By the theorem on the invertibility of perturbed operators, the operator $(A_0 + \Delta A_t)^{-1} \in \mathcal{L}(H,H)$ exists for all $t \in (-t_0, t_0)$. We assume that the value of ε_b' is equal to the value of t_0. Again applying the theorem on the differentiability of the inverse function, we obtain

$$\psi'(\omega) = (\psi_\omega)'(0) = \big((\varphi_w)'(0)\big)^{-1} \in \mathcal{L}(H,H), \qquad w = \psi(\omega)$$

for all $\omega \in G^{\varepsilon_b'}$. Next, we have

$$(A_0 + \Delta A_t)^{-1} = (I + A_0^{-1}\Delta A_t)^{-1} A_0^{-1} = A_0^{-1} + \left(\sum_{n=1}^{\infty} (-1)^n (A_0^{-1}\Delta A_t)^n\right) A_0^{-1}.$$

Hence, by (6), we obtain

$$\|(A_0 + \Delta A_t)^{-1}\| \le \|A_0^{-1}\| + \left(\sum_{n=1}^{\infty} \|A_0^{-1}\|^n \|\Delta A_t\|^n\right) \|A_0^{-1}\| \le K\left(1 + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\right) = 2K.$$

In other words, the estimate $\|\psi'(\omega)\| \le 2K$ holds for all $\omega \in G^{\varepsilon_b'}$.

Now we show that the function $(\psi_{\omega_0})'$ is differentiable (recall that ω_0 is fixed and $w_0 = \psi(\omega_0)$). Since the function $(\varphi_{w_0})'$ is differentiable (on $(\mathcal{U} \times (-\varepsilon_b', \varepsilon_b') - w_0) \cap H$), this function is continuous. The operator function

$$\mathcal{L}(H,H) \to \mathcal{L}(H,H), \qquad A \mapsto A^{-1}$$

is also continuous at any point $A \ne 0$. Since $(\varphi_{w_0})'$ is different from zero on $(\mathcal{U} \times (-\varepsilon_b', \varepsilon_b') - w_0) \cap H$ (invertibility), the function $(\psi_{\omega_0})'$ is continuous on $(G^{\varepsilon_b'} - \omega_0) \cap H$ as a composition of continuous functions. Taking this fact into account, we obtain

$$\begin{aligned}
\psi''(\omega_0)h = (\psi_{\omega_0})''(0)h &= \lim_{t \to 0} \frac{(\psi_{\omega_0})'(th) - (\psi_{\omega_0})'(0)}{t} \\
&= -\lim_{t \to 0} \frac{(\psi_{\omega_0})'(th)((\varphi_{w_0})'(th) - (\varphi_{w_0})'(0))(\psi_{\omega_0})'(0)}{t} \\
&= -(\psi_{\omega_0})'(0)(\varphi_{w_0})'' h (\psi_{\omega_0})'(0) = -\psi'(\omega_0)\varphi''(\psi(\omega_0)) h \psi'(\omega_0), \qquad h \in H.
\end{aligned}$$

For all $\omega \in G^{\varepsilon_b'}$, this implies the estimate

$$\|\psi''(\omega)\| \le \|\psi'(\omega)\|^2 \|\varphi''(\omega)\| \le K^2 K_\varphi. \qquad \square$$

Without loss of generality, from now on we assume that $\varepsilon_b' = \varepsilon_b$.

In the sequel, we need the following theorem.

Theorem 1. *Suppose that* $D_F \subset E$ *is a domain and* $F\colon D_F \to \mathbb{R}^n$ *is a continuous function with the continuous derivative* $F'\colon D_F \to {H'}^n \cong H^n$ *along the subspace* H. *Let* $S = \{\omega \in D_F : F(\omega) = 0\}$, *and let the system of vectors* $F'(\omega)$ *be linearly independent at each point* $\omega \in S$. *Suppose also that* T_b *is a closed subspace of codimension* n *of the space* E, $b = \{b_1, \dots, b_n\}$ *is an orthonormal system of vectors from* $j_H(E')$ *such that the subspace* R_b *is orthogonal in* H *to the subspace* $H_b = T_b \cap H$, *and* $\det\{(n_i^S(\omega), b_j)\} > 0$ *for all* $\omega \in S$ *(here the system of vectors* $n^S(\omega)$ *is obtained from* $F'(\omega)$ *by Schmidt orthogonalization and normalization, and, as above, we identify the space* E *with the space* $T_b \times \mathbb{R}^n$ *by using the isomorphism* $T_b \times \mathbb{R}^n \to E$, $(x, (t_1, \dots, t_n)) \mapsto x + \sum_{i=1}^n t_i b_i$*). Then*

1) *there exists an open (in* T_b*) set* $\mathcal{V} \subset T_b$ *and a continuous function* $g\colon \mathcal{V} \to \mathbb{R}^n$ *such that* $F(x, g(x)) = 0$ *for all* $x \in \mathcal{V}$;
2) *if the function* F *has all derivatives of order* $k \ge 2$ *along the subspace* H, *then the function* g *also has all derivatives of order* $k \ge 2$ *along* H_b;
3) *moreover, if there exist constants* $\lambda > 0$, $\sigma > 0$, *and* $C > 0$ *such that*

$$\|F_i'(\omega)\| > \lambda, \qquad \|F_i'(\omega)\| < C, \qquad \|F_i''(\omega)\| < C, \qquad \det\{(b_i, n_j^S(\omega))\} > \sigma$$

for all $\omega \in S$ *and for all* $0 \le i \le n$, *then there exists a constant* $C_g > 0$ *such that* $\|g_i'(x)\| < C_g$ *and* $\|g_i''(x)\| < C_g$ *for all* $x \in \mathcal{V}$ *and for all* $0 \le i \le n$.

Proof. Assertion 1) holds by virtue of the implicit function theorem [7]. Indeed, the continuous partial derivative $(\partial/\partial t)F(x,t)$ exists at each point $(x,t) \in D_F$; this derivative is different from zero at each point $(x,t) \in S$, because

$$\frac{\partial}{\partial t}F(x,t) = F'(x+tb)b = \|F'(x+tb)\|(n(x+tb), b) > 0.$$

Next, we fix an arbitrary point $\omega = (x_0, g(x_0)) \in S$, where $x_0 \in \mathcal{V}$, and define the functions

$$F_\omega\colon (D_F - \omega) \cap H \to \mathbb{R}, \qquad u \mapsto F(\omega + u),$$
$$g_{x_0}\colon (\mathcal{V} - x_0) \cap H_b \to \mathbb{R}, \qquad x \mapsto g(x_0 + x) - g(x_0).$$

Since $F(x, g(x)) = 0$ for $x \in \mathcal{V}$, we have $F_\omega(x, g_{x_0}(x)) = 0$. The function F_ω satisfies all the assumptions of the theorem on the differentiability of the implicit function g_{x_0} at the point $(0,0)$. Indeed, $(\partial/\partial t)F(x,t)$ is a continuous function $(D_F - \omega) \cap H \to \mathbb{R}$, since

$$\frac{\partial}{\partial t}F_\omega(x,t) = \frac{\partial}{\partial t}F(x + x_0, t + g(x_0)) = F'\bigl(x + x_0 + (t + f(x_0))b\bigr)b. \tag{7}$$

Hence we obtain

$$\frac{\partial}{\partial t}F_\omega(0,0) = \|F'(x_0 + g(x_0)b)\|(n^S(\omega), b) > 0.$$

Finally, the partial derivative

$$\frac{\partial}{\partial x}F_\omega(0,0) = F'(x_0 + g(x_0)b)$$

exists at the point $(0,0)$. Then, by the implicit function theorem [7], the function g_{x_0} has the derivative

$$(g_{x_0})'(0) = -\frac{\partial}{\partial x}F_\omega(0,0)\Big/\frac{\partial}{\partial t}F_\omega(0,0)$$

at the point $x=0$. Taking into account the relation $(g_{x_0})'(0)=g'(x_0)$ and the fact that $x_0\in\mathcal{V}$ is an arbitrary point, we transform the expression in the right-hand side and obtain

$$g'(x)h=\frac{-F'(x+g(x)b)h}{F'(x+g(x)b)b}=\frac{\|F'(x+g(x)b)\|(-n^S(x),h)}{\|F'(x+g(x)b)\|(n^S(x),b)}=\frac{(P_bn^S(x),h)}{(n^S(x),b)} \tag{8}$$

for all $x\in\mathcal{V}$ and for all $h\in H_b$. Hence, as an element of the space H_b, $g'(x)$ can be written in the form

$$g'(x)=\frac{P_bn^S(x)}{(n^S(x),b)}.$$

Assertion 2) readily follows from (8). Moreover, for the second-order derivative of the function g we obtain the expression

$$\begin{aligned}g''(x)hh_1=&-\frac{F''(x+g(x)b)hh_1+(g'(x)h_1)F''(x+g(x)b)hb}{F'(x+f(x)b)b}\\&+\frac{(F'(x+g(x)b)h)F''(x+g(x)b)bh+(g'(x)h_1)F''(x+g(x)b)bb}{(F'(x+g(x)b)b)^2},\end{aligned}$$

from which, using the relation $F'(x+g(x)b)b=\|F'(x+g(x)b)\|(n^S(x),b)$, we obtain assertion 3). $\square$

Remark 6. We could define the surface G using the function F defined on some open set $D_F\subset E$ and satisfying the assumptions of Theorem 1 as the set of points of the form $\{\omega\in D_F: F(\omega)=0\}$ if we assumed that the third-order derivative of the function F exists and is bounded on D_F with respect to the norm of the space $\mathcal{L}(H,\mathcal{L}(H,H))^n$. Then, by Theorem 1, Conditions 1–3 would be satisfied.

2. SURFACE LAYER THEOREMS

A real σ-additive function defined on the σ-algebra $\mathfrak{B}_X$ of Borel subsets of the space X is called a *measure* on the topological space X; the τ_s-differentiability [8] (cf. [9]) of this function means that this *measure is differentiable*.

Let ν be a nonnegative Radon measure defined on E and $n+1$ times differentiable along the subspace R_b, and let ν_G be a surface measure on G defined similarly to [6] and [3] as follows:

$$\nu_G(Q)=\int_Q\frac{\nu^d(dx)}{\det\{(n_i^G(P_bx),b_j)\}}, \tag{9}$$

where

$$\nu^d(Q)=d_{b_1,\dots,b_n}\nu\biggl(\bigcup\biggl\{Q+\sum_{i=1}^n s_ib_i: s_i\le 0,\ 0\le i\le n\biggr\}\biggr),\qquad Q\in\mathfrak{B}_G$$

(here $d_{b_1,\dots,b_n}\nu$ is the nth-order derivative of the measure ν along the directions $b_1,\dots,b_n$).

As in [2], it can be proved that ν_G is a (nonnegative) measure on G.

Let $B\subset G$ be a subset. Since the mapping ψ is continuous for any $\varepsilon\le\varepsilon_b$, the set B^ε is open if the set B is open, and B^ε is a Borel set if B is a Borel set.

For $B\in\mathfrak{B}_G$ and for each $\varepsilon\le\varepsilon_b$, we define the set

$$B_\varepsilon\overset{\text{def}}{=}\biggl\{x+\sum_{i=1}^n t_in_i^G(x): x\in B,\ 0\le i\le n,\ 0\le t_i<\varepsilon\biggr\}, \tag{10}$$

which is a Borel set, since the mapping ψ is continuous. Let ν_T be the projection of the measure ν on T_b along the subspace R_b. We shall say that a *Borel set* $B\subset G$ *has property* $(*)$ if $\nu_T(\partial P_bB)=0$ (the boundary is taken in G). In what follows, $B(\varepsilon)=\{x\in\mathbb{R}^n:\|x\|<\varepsilon\}$ and λ is the Lebesgue measure in $\mathbb{R}^n$.

Theorem 2. *Suppose that $B \subset G$ is a subset open with respect to G, contained in G together with its closure with respect to E, and having property $(*)$. Then*

$$\nu_G(B) = \lim_{\varepsilon \to 0} \frac{\nu(B^\varepsilon)}{\lambda(B(\varepsilon))}. \tag{11}$$

Below, for simplicity, we prove the theorem in the case $n = 1$. The proof of the theorem is based on the following Lemmas 5–10.

Lemma 5. *Suppose that $A \in \mathfrak{B}_\mathcal{U}$, $f_1, f_2 \colon A \to R_b$ are functions, $f_1 \le f_2$, and*

$$A_{f_1,f_2} \overset{\text{def}}{=} \bigl\{\omega \in E : P_b\omega \in A,\ f_1(P_b\omega) \le \omega - P_b\omega \le f_2(P_b\omega)\bigr\}.$$

Suppose also that $Q \subset A_{f_1,f_2}$ is a Borel set. Then the following inequality holds:

$$\nu(Q) \le 4|d_b\nu|(A + R_b)\sup(f_2 - f_1).$$

Proof. This lemma is proved [1] for a separable Banach space. In our case the proof does not change. □

We define the mapping P_G of "projecting on G along the normal" as follows: $P_G \colon G^{\varepsilon_b} \to G$, $x + tn^G(x) \mapsto x$. It is easy to see that $P_G = \varphi(\,\cdot\,, 0) \circ P_b \circ \psi$. Hence the mapping P_G is continuous as a composition of continuous mappings.

We set $G_\varepsilon = \varphi(\mathcal{U} + \varepsilon b)$. By the symbol $L_{\varepsilon,x}$ we denote the set of all vectors tangent to G_ε at the point $x + \varepsilon n^G(x)$, where $x \in \mathcal{U}$, such that for each of these vectors there exists a curve

$$\gamma \colon [0,1] \to G_\varepsilon, \qquad \gamma(0) = x + \varepsilon n^G(x),$$

for which the derivative $\dot\gamma(0)$ exists in the topology of the space H.

In the next lemma, it is convenient to assume that the domain of definition of the mapping $x \mapsto n^G(x)$ is the surface G.

Lemma 6. *The relation $L_{\varepsilon,x} = H_x$ holds for all $x \in G$ and for all $\varepsilon < \varepsilon_b$.*

Proof. We fix an arbitrary point $x \in G$ and a number $\varepsilon < \varepsilon_b$. Without loss of generality, we assume that $x + \varepsilon n^G(x) = 0$. In this case, we simply write L_ε for the set $L_{\varepsilon,x}$. We show that L_ε is a subspace of codimension 1 of the space H.

Let F_b be a continuous linear functional represented by an element b. We define the function

$$\Psi \colon G^{\varepsilon_b} \to \mathbb{R}, \qquad \omega \mapsto F_b(\psi(\omega)). \tag{12}$$

Then the set G_ε can be written as $G_\varepsilon = \{\omega \in G^{\varepsilon_b} : \Psi(\omega) = \varepsilon\}$. We note that the set G^{ε_b} is open, since the mapping ψ is continuous. The derivative $\Psi'(\omega)$ (in the topology of the space H) is nonzero at each point $\omega \in G^{\varepsilon_b}$. Indeed,

$$\Psi'(\omega)h = F_b(\psi'(\omega)h) = (b, \psi'(\omega)h) = \bigl((\psi'(\omega))^*b, h\bigr). \tag{13}$$

But the mapping $(\psi'(\omega))^*$ is injective, because the mapping $\psi'(\omega)$ is bijective. Hence we have $(\psi'(\omega))^*b \ne 0$.

In the space H, we consider the surface $G_\varepsilon \cap H$. Since $\Psi'(\omega)$ is nonzero at each point $\omega \in G^{\varepsilon_b}$, the tangent space at each point of this surface (in particular, at the point $\omega = 0$) is a subspace of codimension 1 in the space H. In particular, L_ε is a subspace of codimension 1 in the space H.

Let us show that $L_\varepsilon = H_x$. Let $\tau \in L_\varepsilon$, and let $\gamma \colon [0,1] \to G_\varepsilon \cap H$ be a curve such that $\gamma(0) = 0$ and $\dot\gamma(0) = \tau$ (the differentiability at the zero point in the topology of the space H). We set $\gamma_1(t) = P_G(\gamma(t))$ and rewrite $\gamma_1(t)$ as $\gamma_1(t) = \varphi(P_b\psi(\gamma(t)), 0)$. This implies that $\gamma_1(t) \in H$ for all $t \in [0, 1]$ and, by Lemma 2, the mapping $\gamma_1(\cdot)$ is differentiable at the point $t = 0$ in the topology of the space H. We have

$$\gamma(t) = \gamma_1(t) + \varepsilon n^G(\gamma_1(t)).$$

Multiplying this relation scalarly by $n^G(x)$, differentiating with respect to t at the point $t = 0$, and using Lemmas 2 and 1, we obtain

$$(\tau, n^G(x)) = (\dot\gamma(0), n^G(x)) = (\dot\gamma_1(0), n^G(x)) + \frac{1}{2}\frac{d}{dt}\|n^G(\gamma_1(t))\|^2\bigg|_{t=0} = 0,$$

i.e., we have proved that $\tau \in H_x$. But both spaces L_ε and H_x are of codimension 1, and hence they coincide. □

Lemma 7. *There exist an $\varepsilon_b' \in (0, \varepsilon_b)$ and a constant $\lambda > 0$ such that, for all $\omega \in G^{\varepsilon_b'}$, the following estimate holds*:

$$|(b, \psi'(\omega)b)| > \lambda. \tag{14}$$

Proof. In the notation of Lemma 4 we have

$$\psi'(x + tn^G(x)) = (A_0 + \Delta A_t)^{-1} = A_0^{-1} - A_0^{-1}\Delta A_t(A_0 + \Delta A_t)^{-1}.$$

It follows from the proof of Lemma 4 that $\|\Delta A_t\| < K_n t$ and $\|(A_0 + \Delta A_t)^{-1}\| < K_\psi$. We have the estimate

$$\big|\big(b, \psi'(x + tn^G(x))b\big)\big| \ge |(b, \psi'(x)b)| - |(b, A_0^{-1}\Delta A_t(A_0 + \Delta A_t)^{-1}b)|. \tag{15}$$

Now we estimate the first term in (15). Let $\psi'(x)b = h$. Then $\varphi'(x, 0)h = b$, or

$$-k(x)(n^G(x), h)b + (b, h)n^G(x) + h = b, \qquad \text{where} \quad k(x) = \sqrt{1 + \|f'(x)\|^2}. \tag{16}$$

It follows from the form of this equation that we must seek its solution in the form $h = \alpha n^G(x) + \beta b$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Substituting this expression in Eq. (16), we obtain the following system of equations for α and β:

$$\begin{cases} \alpha = -\dfrac{1}{k(x) + (1 + p(x))(1 - p(x)k(x))}, \\ \beta = -\alpha(1 + p(x)), \end{cases}$$

where $p(x) = (b, n^G(x))$. As already proved (Lemma 4), $\varphi'(x, 0)$ is a bijection. Hence the obtained solution is unique. We note that

$$(b, \psi'(x)b) = (b, \alpha n^G(x) + \beta b) = \alpha p(x) - \alpha(1 + p(x)) = -\alpha.$$

Taking into account the inequalities $0 < k(x) < \sqrt{1 + K_f^2}$ and $0 < p(x) \le 1$, we obtain the estimate

$$\begin{aligned} \big|k(x) + \big(1 + p(x)\big)\big(1 - p(x)k(x)\big)\big| &\le k(x) + \big(1 + p(x)\big)\big(1 + p(x)k(x)\big) \\ &\le \sqrt{1 + K_f^2} + 2\big(1 + \sqrt{1 + K_f^2}\,\big) \overset{\text{def}}{=} \frac{1}{2\lambda}, \end{aligned}$$

where λ is a constant. Hence we have

$$|(b, \psi'(x)b)| > 2\lambda. \tag{17}$$

Now we estimate the second term in (15):

$$|(b, A_0^{-1}\Delta A_t(A_0 + \Delta A_t)^{-1}b)| \le \|A_0^{-1}\| \cdot \|\Delta A_t\| \cdot \|(A_0 + \Delta A_t)^{-1}\| \le K_\psi^2 K_n t.$$

We choose a $t_0 > 0$ such that $K_\psi^2 K_n t_0 < \lambda$. Then, taking into account (17), we see that (14) holds for all $0 < t < t_0$. We set ε_b' to be equal to this value t_0. □

Without loss of generality, we assume that $\varepsilon_b' = \varepsilon_b$.

Lemma 8. *The set* G_ε *is the graph of a continuous function* $g_\varepsilon\colon \mathcal{U}(\varepsilon) \to \mathbb{R}$ *defined on some set* $\mathcal{U}(\varepsilon)$ *open in* T_b *and having the first-order derivative* $g'_\varepsilon\colon \mathcal{U}(\varepsilon) \to H' \cong H$ *and the second-order derivative* $g''_\varepsilon\colon \mathcal{U}(\varepsilon) \to \mathcal{L}(H, \mathcal{L}(H, H))$ *along the subspace* H. *Moreover, there exists a constant* $K_g > 0$ *such that* $\|g''_\varepsilon(x)\| < K_g$ *for all* $x \in \mathcal{U}(\varepsilon)$.

Proof. Suppose that, as before, the function Ψ is defined by formula (12). We apply Theorem 1 to the function $\Psi - \varepsilon$. Earlier it was proved that the mapping ψ' is continuous, and hence the mapping Ψ' is also continuous. This implies that the derivative $\Psi'(\omega)$ is different from zero at each point $\omega \in G^{\varepsilon_b}$. Next, by Lemma 6, we obtain $n^{G^{\varepsilon_b}}(\omega) = n^G(x)$, where $x = P_b P_G(\omega)$. Therefore, by Remark 3, we have $(n^{G^{\varepsilon_b}}(\omega), b) > 0$ for all $\omega \in G^{\varepsilon_b}$. By Theorem 1, there exists a continuous function g_ε defined on some open set $\mathcal{U}(\varepsilon) \subset T_b$, having first- and second-order derivatives along the subspace H, and satisfying the condition that G_ε is the graph of this function (the fact that the function Ψ also has a second-order derivative along the subspace H follows from formula (13)).

Now we verify condition 3) of Theorem 1, which implies that $\|g''_\varepsilon(x)\|$ is bounded on $\mathcal{U}(\varepsilon)$ by a constant K_g. By Remark 3 and Lemma 6, for all $\omega \in G^{\varepsilon_b}$, we have

$$(b, n^{G^{\varepsilon_b}}(\omega)) > \frac{1}{\sqrt{1 + K_f^2}} > 0.$$

Next, by (13) and Lemma 4, we have $\|\Psi''(\omega)\| < K_\psi$ for all $\omega \in G^{\varepsilon_b}$.

Finally, we show that there exists a constant $\lambda > 0$ such that $\|\Psi'(\omega)\| > \lambda$. Because we have

$$|(b, n^{G^{\varepsilon_b}}(\omega))| \le 1 \qquad \text{and} \qquad \|\Psi'(\omega)\| \cdot |(b, n^{G^{\varepsilon_b}}(\omega))| = |\Psi'(\omega)b|,$$

it suffices to prove that $|\Psi'(\omega)b| > \lambda$. But

$$\Psi'(\omega)b = F_b(\psi'(\omega)b) = (b, \psi'(\omega)b),$$

and the statement of this lemma follows from Lemma 7. □

Lemma 9. *On the set* $\mathcal{U} \cap \mathcal{U}(\varepsilon)$, *the function* g_ε *has the form*

$$g_\varepsilon(x) = f(x) + \frac{\varepsilon}{(n^G(x), b)} + \varepsilon^2 \alpha_\varepsilon(x);$$

moreover, there exists a $C > 0$ *such that* $|\alpha_\varepsilon(x)| < C$ *for all* $\varepsilon \in (0, \varepsilon_b)$.

Proof. We choose an arbitrary point $x_0 \in \mathcal{U} \cap \mathcal{U}(\varepsilon)$. We set $x_1 = P_b(g_f(x_0) + \varepsilon n^G(x_0))$ and consider the restriction of the function g_ε to the straight line passing through the points x_0 and x_1. We define the function

$$\widetilde{g}_\varepsilon(t) = g_\varepsilon\biggl(x_1 + t\frac{x_0 - x_1}{\|x_0 - x_1\|}\biggr).$$

We set $t_0 \overset{\text{def}}{=} \|x_0 - x_1\|$. Applying the Taylor formula to the function $\widetilde{g}_\varepsilon$, we obtain

$$\widetilde{g}_\varepsilon(t_0) = \widetilde{g}_\varepsilon(0) + \widetilde{g}'_\varepsilon(0)t_0 + \frac{1}{2}\widetilde{g}''_\varepsilon(\theta t_0)t_0^2,$$

where $\theta \in (0, 1)$.

Through the point $(x_1, g_\varepsilon(x_1))$, we draw the tangent $k(t) = \widetilde{g}_\varepsilon(0) + \widetilde{g}'_\varepsilon(0)t$ to the graph of the function $\widetilde{g}_\varepsilon$. By Lemma 6, we have

$$k(t_0) - f(x_0) = \frac{\varepsilon}{(b, n^G(x_0))}. \tag{18}$$

Next, taking into account the relation $\widetilde{g}_\varepsilon(t_0) = g_\varepsilon(x_0)$, we obtain

$$g_\varepsilon(x_0) - f(x_0) = \bigl(k(t_0) - f(x_0)\bigr) + \bigl(g_\varepsilon(x_0) - k(t_0)\bigr), \tag{19}$$

where

$$|g_\varepsilon(x_0) - k(t_0)| = \left|\frac{1}{2}\widetilde{g}''_\varepsilon(\theta t_0)\|x_0 - x_1\|^2\right| \le \frac{1}{2}K_g\varepsilon^2.$$

This, as well as (18) and (19), implies the statement of the lemma. □

Lemma 10. *Suppose that $B \subset G$ is a Borel set contained in G together with its closure. Then there exists a constant $K > 0$ such that, for any $x \in T_b$ and for any $\varepsilon < \varepsilon_b$, the set $(x+R_b) \cap \overline{B}_\varepsilon$ ($\overline{B}_\varepsilon$ is the closure of B_ε) is contained in an interval of the line $x+R_b$ of length less than $K\varepsilon$.*

Proof. It is clear that it suffices to prove the statement of the lemma for a closed set B. We set

$$G_{\varepsilon,B} \stackrel{\text{def}}{=} \varphi(P_b B, \varepsilon).$$

The line $x + R_b$ has a nonempty intersection with $\overline{B}_\varepsilon$ if and only if $x \in P_b G_{\varepsilon,B} \cup P_b B$. If $x \in P_b G_{\varepsilon,B} \cap P_b B$, then the statement of the lemma follows from Lemma 9.

Let $x \in P_b B \setminus P_b G_{\varepsilon,B}$ (if the set $P_b B \setminus P_b G_{\varepsilon,B}$ is empty, then we do not consider this case). The subset $(x + R_b) \cap \overline{B}_\varepsilon$ of the number line $x + R_b$ is closed and bounded; hence it contains its least upper bound $\overline{\omega}$. It is clear that

$$\overline{t} \stackrel{\text{def}}{=} \|P_G(\overline{\omega}) - \overline{\omega}\| < \varepsilon.$$

The intersection of the set $\overline{B}_\varepsilon$ with the line $x+R_b$ is the interval $[\varphi(x,0), \overline{\omega}]$. Its length is equal to $g_{\overline{t}}(x) - f(x)$. The statement of the lemma again follows from Lemma 9 applied to the function $g_{\overline{t}}$.

Now let $x \in P_b G_{\varepsilon,B} \setminus P_b B$, and let $\overline{\omega} \stackrel{\text{def}}{=} \inf\{(x+R_b) \cap \overline{B}_\varepsilon\}$ (if the set $P_b G_{\varepsilon,B} \setminus P_b B$ is empty, then the lemma is proved). The intersection of the set $\overline{B}_\varepsilon$ with the line $x + R_b$ is the interval $[\overline{\omega}, x + g_\varepsilon(x)b]$. Let us estimate its length. Let

$$y \stackrel{\text{def}}{=} P_b\big(P_G(\overline{\omega}) + \varepsilon n^G(P_b P_G(\overline{\omega}))\big).$$

We set

$$e = \frac{x-y}{\|x-y\|}$$

and define the function $g_e(t) = g_\varepsilon(y+te)$, $t \in \mathbb{R}$. We also set $t_x \stackrel{\text{def}}{=} \|x-y\|$. Let k_e be the tangent to the graph of the function $g_e(t)$ at the point $t = 0$. By Lemma 6, the tangent k_e is orthogonal to the vector $n^G(P_b P_G(\overline{\omega}))$. Then, taking into account the inequality $(b, n^G(x)) > 1/\sqrt{1+K_f^2}$, we obtain the estimate

$$\|x + k_e(t_x)b - \overline{\omega}\| = \frac{\|\overline{\omega} - y + g_\varepsilon(y)b\|}{(b, n^G(x))} < K_1\varepsilon, \tag{20}$$

where K_1 is an appropriate constant. By the Taylor formula with remainder in Lagrange form, there exists a $t_0 \in (0, t_x)$ such that

$$|g_e(t_x) - k_e(t_x)| = \left|\frac{1}{2} g_e''(t_0) t_x^2\right| \le \frac{1}{2} K_g \varepsilon^2.$$

Hence, by (20) we have

$$\|x + g_\varepsilon(x)b - \overline{\omega}\| \le \|x + k_e(t_x)b - \overline{\omega}\| + |g_e(t_x) - k_e(t_x)| < K\varepsilon,$$

where K is an appropriate constant; we also took into account the fact that $g_e(t_x) = g_\varepsilon(x)$. The proof of the lemma is complete. □

Proof of Theorem 2. On the set $\mathcal{U}\times\mathbb{R}$, we define the function

$$f(x,\varepsilon)=f(x)+\frac{\varepsilon}{(n^G(x),b)}.$$

Let $B\subset W_b$ be an open set with property $(*)$. We set $f_\varepsilon(x)=f(x,\varepsilon)$ and

$$B_{f,f_\varepsilon}\overset{\text{def}}{=}\{x+tb,\ x\in P_bB,\ f(x)\le t<f_\varepsilon(x)\}. \tag{21}$$

We apply the results of [1] (obviously, the conditions under which these results hold are satisfied) and obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\nu(B_{f,f_\varepsilon})=\int_B(n^G(x),b)\frac{\partial f}{\partial\varepsilon}(x,0)\,\nu_G(dx)=\nu_G(B). \tag{22}$$

Here the first relation holds because of the results in [1]. Let B_ε be a set of the form (10). If we prove that

$$\nu(B_\varepsilon\bigtriangleup B_{f,f_\varepsilon})=o(\varepsilon) \tag{23}$$

for $\varepsilon<\varepsilon_b$, then we obtain the relation

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\nu(B_\varepsilon)=\nu_G(B).$$

Changing the normal vector field to the opposite and taking into account the fact that Lemma 5 implies $\nu(B)=0$, we prove (11). Now we prove (23). Let $G_{\varepsilon,B}\overset{\text{def}}{=}\varphi(P_bB,\varepsilon)$. We set

$$M_1\overset{\text{def}}{=}(B_\varepsilon\bigtriangleup B_{f,f_\varepsilon})\cap(P_bB\cap P_bG_{\varepsilon,B}+R_b),$$
$$M_2\overset{\text{def}}{=}(B_\varepsilon\bigtriangleup B_{f,f_\varepsilon})\cap(P_bB\setminus P_bG_{\varepsilon,B}+R_b),$$
$$M_3\overset{\text{def}}{=}(B_\varepsilon\bigtriangleup B_{f,f_\varepsilon})\cap(P_bG_{\varepsilon,B}\setminus P_bB+R_b).$$

Obviously, $M_1\cup M_2\cup M_3=B_\varepsilon\bigtriangleup B_{f,f_\varepsilon}$. The set $G_{\varepsilon,B}$ is open (in G_ε) as the preimage of the open (in $T_b+\varepsilon b$) set $P_bB+\varepsilon b$ under the continuous mapping ψ. Since the function g_ε is continuous, the set $P_bG_{\varepsilon,B}$ is open (in T_b); hence each of the sets M_1, M_2, and M_3 is a Borel set. Let us estimate the measure of each of them. To estimate $\nu(M_1)$, we note that M_1 has the form

$$M_1=\bigl\{x+tb: x\in P_bB\cap P_bG_{\varepsilon,B},\ \min\{f_\varepsilon(x),\ g_\varepsilon(x)\}\le t<\max\{f_\varepsilon(x),\ g_\varepsilon(x)\}\bigr\}$$

(the function g_ε is defined in Lemma 9). By Lemma 9, we have

$$\max\{f_\varepsilon(x),\ g_\varepsilon(x)\}-\min\{f_\varepsilon(x),\ g_\varepsilon(x)\}<C\varepsilon^2,$$

where C is a constant. Using Lemma 5, we obtain

$$\nu(M_1)\le 4|d_b\nu|(P_bB\cap P_bG_{\varepsilon,B}+R_b)C\varepsilon^2\le 4|d_b\nu|(P_bB+R_b)C\varepsilon^2. \tag{24}$$

Now, let us estimate $\nu(M_2)$. We set

$$M_{21}\overset{\text{def}}{=}(B_{f,f_\varepsilon}\setminus B_\varepsilon)\cap(P_bB\setminus P_bG_{\varepsilon,B}+R_b),\qquad M_{22}\overset{\text{def}}{=}(B_\varepsilon\setminus B_{f,f_\varepsilon})\cap(P_bB\setminus P_bG_{\varepsilon,B}+R_b).$$

Clearly, the sets M_{21} and M_{22} are Borel sets, and $M_{21}\cup M_{22}=M_2$. First, we estimate $\nu(M_{21})$. Taking into account the relations $M_{21}\subset B_{f,f_\varepsilon}$ and $\sup_x|f_\varepsilon(x)-f(x)|<C_1\varepsilon$ (where C_1 is an appropriate constant) and using Lemma 5, we obtain

$$\nu(M_{21})\le 4C_1\varepsilon|d_b\nu|(P_bB\setminus P_bG_{\varepsilon,B}+R_b). \tag{25}$$

Now we estimate $\nu(M_{22})$. For $x \in P_bB \setminus P_bG_{\varepsilon,B}$, we consider the subset $(x+R_b)\cap \overline{B}_\varepsilon$ of the number line $x+R_b$. It is closed and bounded; hence it contains its least upper bound. On $P_bB \setminus P_bG_{\varepsilon,B}$ we define the function

$$h_\varepsilon(x) = f(x) + \sup\{(x+R_b)\cap \overline{B}_\varepsilon\}.$$

By Lemma 10, we have

$$\sup_x |h_\varepsilon(x) - f(x)| < K\varepsilon$$

for all $\varepsilon < \varepsilon_b$. Then, taking into account $M_{22} \subset B_\varepsilon$ and using Lemma 5, we obtain

$$\nu(M_{22}) \le 4K\varepsilon |d_b\nu|(P_bB \setminus P_bG_{\varepsilon,B} + R_b). \tag{26}$$

Let us show that

$$|d_b\nu|(P_bB \setminus P_bG_{\varepsilon,B} + R_b) \to 0, \qquad \varepsilon \to 0. \tag{27}$$

To this end, it suffices to prove that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_bB\setminus P_bG_{\varepsilon,B}}(x) = 0, \qquad x \in T_{x_0} \tag{28}$$

($\mathbb{I}$ is the indicator function of the set in the subscript). Indeed, it follows from (28) that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{(P_bB\setminus P_bG_{\varepsilon,B})+R_b}(x) = 0, \qquad x \in E,$$

which, by the Lebesgue theorem, implies (27).

To prove (28), it suffices to show that, for any point $x \in P_bB$, there exists an $\overline{\varepsilon} > 0$ such that $x \in P_bG_{\varepsilon,B}$ for all $\varepsilon < \overline{\varepsilon}$. We choose an arbitrary point $x \in P_bB$. The set $(x+R_b)\cap B^\varepsilon$ is an open subset of the number line $x+R_b$. On the line $x+R_b$, we choose a neighborhood U_{ω_0} of the point $\omega_0 = x + f(x)b$ so that it is contained in $(x+R_b)\cap B^\varepsilon$. We set

$$\omega(t) = x + (f(x)+t)b, \qquad t \in \mathbb{R}.$$

Let $\overline{t} > 0$, so that $\omega(\overline{t}\,) \in U_{\omega_0}$. On the interval $[0,\overline{t}\,]$, we define the function

$$\xi(t) = \|\omega(t) - P_G(\omega(t))\|.$$

The function $\xi(t)$ is continuous on the interval $[0,\overline{t}\,]$. Indeed, the function ξ has the form

$$\xi(t) = \|P_b\psi(\omega(t)) - \psi(\omega(t))\|,$$

i.e., is the composition of the two functions $\psi(\omega(t))$ and $k(\omega) = \|P_b\omega - \omega\|$. The function $\psi(\omega(t))$ is continuous as the composition of continuous functions. Let us show that $k(\omega)$ is continuous. We write the argument ω in the form $\omega = (z,t)$, where $z \in T_b$ and $t \in \mathbb{R}$ ($E \simeq T_b \times \mathbb{R}$). Then the function k can be rewritten as $k\colon (z,t) \mapsto t$. Now it is obvious that this function is continuous. Hence the function $\xi(t)$ is continuous as a composition of continuous functions. We choose the value $\xi(\overline{t}\,)$ to be $\overline{\varepsilon}$. Clearly, we obtain $x \in P_bG_{\overline{\varepsilon},B}$. But on the interval $[0,\overline{t}\,]$, the continuous function $\xi(t)$ takes all intermediate values from 0 to $\overline{\varepsilon}$. Hence $x \in P_{x_0}G_{\varepsilon,B}$ for all $\varepsilon < \overline{\varepsilon}$. Relation (28) is thus proved.

Now let us estimate $\nu(M_3)$. For $x \in P_bG_{\varepsilon,B} \setminus P_bB$, we define the function

$$h_{1\varepsilon}(x) = g_\varepsilon(x) - \inf\{(x+R_b)\cap \overline{B}_\varepsilon\}.$$

It follows from Lemma 10 that

$$\sup_x |h_{1\varepsilon}(x) - g_\varepsilon(x)| < K\varepsilon$$

for all $\varepsilon < \varepsilon_b$. Using Lemma 5, we obtain

$$\nu(M_3) \le 4K\varepsilon |d_b\nu|(P_b G_{\varepsilon,B} \setminus P_b B + R_b). \tag{29}$$

Let us show that

$$|d_b\nu|(P_b G_{\varepsilon,B} \setminus P_b B + R_b) \to 0, \qquad \varepsilon \to 0. \tag{30}$$

We set $B_1 = \mathcal{U} \setminus \overline{B}$ (here $\overline{B}$ is the closure of B). The set B_1 is open (in G). The foregoing argument shows that, at each point $x \in T_b$, we have

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b B_1 \setminus P_b G_{\varepsilon,B_1}}(x) = 0;$$

hence

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b B_1 \setminus P_b \overline{G}_{\varepsilon,B_1}}(x) = 0, \qquad x \in T_b.$$

Then, taking into account the inclusion $(P_b G_{\varepsilon,B} \cap \mathcal{U}) \setminus P_b\overline{B} \subset P_b B_1 \setminus P_b \overline{G}_{\varepsilon,B_1}$, we obtain

$$\lim_{\varepsilon\to 0} \mathbb{I}_{(P_b G_{\varepsilon,B} \cap \mathcal{U}) \setminus P_b\overline{B}} = 0. \tag{31}$$

Next, let us show that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b G_{\varepsilon,B} \setminus P_b\overline{B}} = 0. \tag{32}$$

To this end, by (31), it suffice to prove that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b G_{\varepsilon,B} \setminus \mathcal{U}}(x) = 0 \tag{33}$$

everywhere on T_b. We choose an arbitrary point $x_0 \in T_b \setminus \mathcal{U}$. We assume that the convergence (33) does not occur at the point x_0. Then there exists a sequence $\varepsilon_n \to 0$ such that $x_0 \in P_b G_{\varepsilon_n,B} \setminus \mathcal{U}$ for all n. Without loss of generality, we assume that $x_0 \in H$. We set $G^H_{\varepsilon_n,B} = G_{\varepsilon_n,B} \cap H$ and $\mathcal{U}^H = \mathcal{U} \cap H$ and obtain

$$x_0 \in P_b G^H_{\varepsilon_n,B} \setminus \mathcal{U}^H. \tag{34}$$

On the other hand, since $\mathcal{U}^H$ is open with respect to the induced topology of the space H_b, it is open in its Hilbert topology. Similarly, the set $\overline{B}^H = \overline{B} \cap H$ is closed in the Hilbert space H_b. We note that the set $P_b G^H_{\varepsilon_n,B}$ is contained in the ε_n-neighborhood of the set $\overline{B}^H$, and, because we have $\overline{B}^H \subset \mathcal{U}^H$, all such ε_n-neighborhoods, starting from some n, are contained in $\mathcal{U}^H$, which contradicts (34). So relation (33) and hence (32) hold.

The indicator functions of arbitrary sets A_1 and A_2 satisfy the relation

$$\mathbb{I}_{A_1 \setminus A_2} = \mathbb{I}_{A_1} - \mathbb{I}_{A_2} + \mathbb{I}_{A_2 \setminus A_1}. \tag{35}$$

We have $P_b G_{\varepsilon,B} \setminus P_b\overline{B} = (P_b G_{\varepsilon,B} \setminus P_b B) \setminus P_b \partial B$. Applying (35) to the sets $P_b G_{\varepsilon,B} \setminus P_b B$ and $P_b \partial B$, we obtain

$$\mathbb{I}_{P_b G_{\varepsilon,B} \setminus P_b\overline{B}} = \mathbb{I}_{P_b G_{\varepsilon,B} \setminus P_b B} - \mathbb{I}_{P_b \partial B} + \mathbb{I}_{P_b \partial B \setminus (P_b G_{\varepsilon,B} \setminus P_b B)}.$$

Hence

$$\mathbb{I}_{P_b G_{\varepsilon,B} \setminus P_b\overline{B} + R_b} = \mathbb{I}_{P_b G_{\varepsilon,B} \setminus P_b B + R_b} - \mathbb{I}_{P_b \partial B + R_b} + \mathbb{I}_{P_b \partial B \setminus (P_b G_{\varepsilon,B} \setminus P_b B) + R_b}. \tag{36}$$

Since $\nu_T(P_b\partial B)=0$ (property $(*)$), we have $\nu(\partial B+R_b)=0$; hence

$$|d_b\nu|(\partial B+R_b)=0. \tag{37}$$

It follows from (36) that

$$\begin{aligned}|d_b\nu|(P_bG_{\varepsilon,B}\setminus P_b\overline{B}+R_b)&=|d_b\nu|(P_bG_{\varepsilon,B}\setminus P_bB+R_b)-|d_b\nu|(\partial B+R_b)\\&\quad+|d_b\nu|\big((\partial B+R_b)\setminus((P_bG_{\varepsilon,B}\setminus P_bB)+R_b)\big),\end{aligned}$$

where, by (37), the last two terms are zero. The fact that the measure $|d_b\nu|$ is σ-additive, the Lebesgue theorem, and relation (32) imply (30).

Now it is clear that (23) follows from the estimates (24)–(26) and (29), as well as from (27) and (30). □

Next, we assume that the function f can be continued to some domain $\mathcal{U}_1\subset T_b$ such that $\overline{\mathcal{U}}\subset\mathcal{U}_1$ and the graph of the function continued is a surface in the sense of Definition 2. By the symbol Σ we denote the system of all open subsets of the surface G that possess property $(*)$.

Next, let ν^ε be the measure on G defined by the relation

$$\nu^\varepsilon(B)=\frac{\nu(B^\varepsilon)}{\lambda(B(\varepsilon))}.$$

In the case of an arbitrary finite codimension n, the operator P_G is defined as follows:

$$P_G\colon G^{\varepsilon_b}\to G,\qquad x+\sum_{i=1}^n t_in_i^G(x)\mapsto x.$$

Theorem 3. *Suppose the family of functions $f_B(\varepsilon)=\nu^\varepsilon(B)/\nu_G(B)$, where $B\in\Sigma$, is uniformly bounded on the half-interval $(0,\varepsilon_b]$. Then formula (11) holds for any Borel subset of the surface G.*

Proof. It follows from the assumptions imposed on the function f that relation (11) holds for open subsets of the surface G that have property $(*)$.

Now let $B\subset G$ be an arbitrary open set. We use the fact that ν is a Radon measure. Similarly to the case $n=1$ studied in [6] and in [2], it can be proved that ν_G is also a Radon measure on G. We shall find a σ-compact set $K_1\subset B$ and a σ-compact set $\widetilde{K}_2\subset B^{\varepsilon_b}$ such that $\nu_G(K_1)=\nu_G(B)$ and $\nu(\widetilde{K}_2)=\nu(B^{\varepsilon_b})$. Let $K_2=P_G(\widetilde{K}_2)$. Since the mapping P_G is continuous, K_2 is a σ-compact set. We set $K=K_1\cup K_2$. It is clear that K is a σ-compact set, $K\subset B$, $\nu_G(K)=\nu_G(B)$, and $\nu(K^{\varepsilon_b})=\nu(B^{\varepsilon_b})$. For each point $x\in K$, we choose its neighborhood $\mathcal{V}_x\subset B$ which has property $(*)$ and satisfies the conditions $\nu^{\varepsilon_b}(\partial\mathcal{V}_x)=0$ and $\nu_G(\partial\mathcal{V}_x)=0$ (we take into account the fact that the measures ν^{ε_b} and ν_G are σ-additive on G). From the open covering of the σ-compact set K by the sets $\mathcal{V}_x$, we choose a countable subcovering $\mathcal{V}_i$, $i=1,2,\dots$. We set

$$\mathcal{F}_1=\mathcal{V}_1,\qquad \mathcal{F}_i=\mathcal{V}_i\setminus\overline{\bigcup_{k=1}^{i-1}\mathcal{V}_k},\quad i=2,3,\dots.$$

The sets $\mathcal{F}_i$ are open and have property $(*)$. Hence, for all sets $\mathcal{F}_i$, $i=1,2,\dots$, relation (11) holds. We set $\mathcal{F}=\bigcup_i\mathcal{F}_i$ and show that (11) also holds for the set $\mathcal{F}$. Since the sets $\mathcal{F}_i$ do not intersect pairwise, we have

$$\lim_{\varepsilon\to0}\frac{\nu(\mathcal{F}^\varepsilon)}{\lambda(B(\varepsilon))}=\lim_{\varepsilon\to0}\frac{1}{\lambda(B(\varepsilon))}\sum_{i=1}^\infty\nu(\mathcal{F}_i^\varepsilon)=\lim_{\varepsilon\to0}\sum_{i=1}^\infty\nu^\varepsilon(\mathcal{F}_i). \tag{38}$$

It follows from the assumptions of the theorem that the series $\sum_{i=1}^{\infty} \nu^{\varepsilon}(\mathcal{F}_i)$ can be majorized by the converging numerical series $\sum_{i=1}^{\infty} \nu_G(\mathcal{F}_i)$ multiplied by some positive constant and hence converges uniformly. Then we can pass to the limit under the sum sign in (38). Taking into account the σ-additivity of the measure ν_G, we obtain

$$\nu_G(\mathcal{F}) = \lim_{\varepsilon \to 0} \frac{\nu(\mathcal{F}^{\varepsilon})}{\lambda(B(\varepsilon))}.$$

The sets $\mathcal{F}_i$ satisfy the conditions $\nu^{\varepsilon_b}(\partial \mathcal{F}_i) = 0$ and $\nu_G(\partial \mathcal{F}_i) = 0$. Thus for any $\varepsilon \le \varepsilon_b$, the set $K^{\varepsilon} \setminus \mathcal{F}^{\varepsilon}$ has the zero ν-measure, and the set $K \setminus \mathcal{F}$ has the zero ν_G-measure. Hence, for all $\varepsilon \le \varepsilon_b$, we have $\nu(\mathcal{F}^{\varepsilon}) = \nu(B^{\varepsilon})$ and $\nu_G(\mathcal{F}) = \nu_G(B)$. Then

$$\nu_G(B) = \nu_G(\mathcal{F}) = \lim_{\varepsilon \to 0} \frac{\nu(\mathcal{F}^{\varepsilon})}{\lambda(B(\varepsilon))} = \lim_{\varepsilon \to 0} \frac{\nu(B^{\varepsilon})}{\lambda(B(\varepsilon))}.$$

We have proved (11) for open subsets of the surface G. Now let $B \subset G$ be a closed set. Because relation (11) holds for the surface G itself, which is an open subset of the surface G_1, and for its open subset $G \setminus B$, this relation also holds for the set B.

Now suppose that $B \subset G$ is an arbitrary Borel set; K_n is a sequence of compact sets such that

$$K_n \subset B \quad \text{and} \quad \nu_G(K_n) \to \nu_G(B) \qquad n \to \infty,$$

and A_n is a sequence of open sets such that $B \subset A_n$ and $\nu_G(A_n) \to \nu_G(B)$ $(n \to \infty)$. Both sequences exists because ν_G is a Radon measure. For each chosen n, we have

$$\nu_G(K_n) = \lim_{\varepsilon \to 0} \frac{\nu(K_n^{\varepsilon})}{\lambda(B(\varepsilon))} \le \varliminf_{\varepsilon \to 0} \frac{\nu(B^{\varepsilon})}{\lambda(B(\varepsilon))} \le \varlimsup_{\varepsilon \to 0} \frac{\nu(B^{\varepsilon})}{\lambda(B(\varepsilon))} \le \lim_{\varepsilon \to 0} \frac{\nu(A_n^{\varepsilon})}{\lambda(B(\varepsilon))} = \nu_G(A_n).$$

Passing to the limit in this system of inequalities as $n \to \infty$, we obtain (11). $\square$

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