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# SURFACE MEASURES AND THE STOKES FORMULA IN LOCALLY CONVEX SPACES

(specialization - 01.01.01 - mathematical analysis)

DISSERTATION
submitted for the academic degree
of candidate of physical and mathematical sciences

Scientific supervisor: doctor
of physical and mathematical sciences,
Professor O.G. Smolyanov

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# Contents

# Abstract

## General description of the thesis

### Relevance of the topic

The Ph.D. thesis belongs to infinite dimensional analysis. Two classes of surface measure in locally convex spaces are considered. The first class consists of surface measures on (infinite dimensional) manifolds of locally convex spaces possessing a finite codimension. Here, it is assumed that surface measures are generated by smooth measures on the ambient spaces. The second class consists of surface measures on (infinite dimensional) manifolds possessing an infinite codimension. Here, as an ambient space, we consider the space of continuous factions defined on a square and taking values in a Euclidean space. It is assumed that on the ambient spaces, the measure generated by a so called Brownian sheet is given. As a submanifold, we consider the totality of continuous functions defined on the same square and taking values in a compact Riemannian manifold of this Euclidean space. We also prove an analog of Chernoff's theorem for evolution families of operators.

Studying of surface measures from the first class is one of the traditional trends of infinite dimensional analysis. It closely related to investigation of finite dimensional differential operators and the general problem of desintegration of measures. Studying of these surface measures has started in works of A. V. Skorokhod [24] and A. V. Uglanov [36] about 30 years ago within the theory of smooth measures on infinite dimensional spaces created in works of S. V. Fomin, O. G Smolyanov, and their students. The theory of surface measures is essentially used in the so called Malliavin calculus [18], [4]. Nowadays, this area of infinite dimensional analysis can be considered as classical.

The technique developed for investigation of surface measures on submanifolds of a finite codimension, nevertheless, turned out to be insufficient for studying of surface measures on submanifolds possessing simultaneously an infinite dimension and an infinite codimension. The appeared difficulties were overcome in the series of works of O. G. Smolyanov, H. v. Weizsäcker, and their coauthors [27], [25], [30], [21]. In these works, the technique of construction of surface measures on submanifolds of a vector space of $\mathbb{R}^n$-valued functions of one real variable was developed. It was assumed, that the manifolds consist of functions taking values in a Rimannian submanifold of $\mathbb{R}^n$. The obtained results are related to studying evolutional differential equations on manifolds. The next natural step is generalizing this technique to the case of a vector space and its submanifold consisting of functions of several real variables [31] (see also [11], [12], [18], [4], [33]). Manifolds of this type appear in quantum field theory and $M$-theory. Thus, the topic of the Ph.D. thesis appears to be quite relevant.

### Scientific novelty

All the results of the Ph.D. thesis are new. The main results are the following:

1. A method of approximation of Uglanov's surface measures by measures of some neighborhoods of submanifolds of codimension 1 in a locally convex space; the surface layer theorem.

2. A calculus of differential forms of a finite codegree in a locally convex space; the Stokes formula for surfaces of codimension 1 in a locally convex space.

3. An analog of Chernoff's theorem for evolutional families of operators.

4. A method of construction of a Brownian sheet with values in a compact Riemannian manifold embedded into a finite dimensional Euclidean space. This result essentially generalizes an analogous result of P. Malliavin for Lie groups.

### Investigation methods

In the Ph.D. thesis we use the methods of infinite dimensional and stochastic analysis, as well as some special constructions.

### Theoretical and practical value

The thesis has a theoretical perspective. Its results can be used for solution of problems of stochastic analysis on manifolds. In particular, for investigation of stochastic fields with values in a compact Riemannian manifold.

### Approbation of the Ph.D. thesis

The results of the Ph.D. thesis were presented at the seminar "Infinite dimensional analysis and mathematical physics" headed by Professor O. G. Smolyanov and Professor E. T. Shavgulidze at the Department of Mechanics and Mathematics of MSU, at the seminar of academician V. S. Vladimirov and corresponding member of Russian Academy of Sciences I. V. Volovich at the department of mathematical physics of Steklov Mathematical Institute of Russian Academy of Sciences, and at the XXV conference "Lomonosov readings" at MSU (2003).

### Publication

The main contents of the Ph.D. thesis has been published in 3 papers of the author; the author does not have papers in coauthorship.

### Organization and amount of work

The thesis consists of the abstract and 3 main chapters divided into paragraphs. Number of pages is 108. The list of references contains 40 items.

## Brief contents of the thesis

### Abstract

In the abstract, we formulate the main results of the thesis and give a review of works on the topic of the thesis.

### Chapter 1

In this chapter, we consider surface measures on submanifolds of a finite codimension and prove the surface layer theorems. Everywhere in this chapter, $E$ is a locally convex space, $H$ its Hilbert subspace, $H$ is dense in $E$, and the identical embedding of $H$ into $E$ is continuous.

DEFINITION 1. *Let $D_F \subset E$ be an open set and $F : D_F \to \mathbb{R}$ be a continuous function having a derivative $F' : D_F \to H' \cong H$ along the subspace $H$. Let $S \stackrel{\text{def}}{=} \{x \in D_F : F(x) = 0\}$, and $F'(x) \neq 0$ for all $x \in S$. We call the vector $n^s = \frac{F'(x)}{\|F'(x)\|}$ (we identify the linear continuous on $H$ functional $F'(x)$ with an element of $H$) the normal vector to $S$ at the point $x$.*

Let the set $S$ be such as in Definition 1, and $B \subset S$ be a subset. For all $\varepsilon > 0$, we define a set

$$B^\varepsilon \stackrel{\text{def}}{=} \{x + tn^s(x) \ : \ x \in B \ , \ t \in (-\varepsilon, \varepsilon)\} \ .$$

DEFINITION 2. *We call the set $G$ a surface, if the following assumptions are fulfilled (1-3):*
ASSUMPTION 1. *The set $G$ is a graph of a continuous bounded function $f$, defined on an open subset $\mathcal{U}$ of a closed space $T_b$ of codimension 1 of the space $E$ and taking values on the line $R_b = \{tb : t \in \mathbb{R}\}$, where $b \in H$ is a unit vector orthogonal in $H$ to the subspace $H_b \stackrel{\text{def}}{=} T_b \bigcap H$.*

Remark 1. We denote by $P_b$ the projector operator on the subspace $T_b$ along the vector $b$.

Assumption 2. *The function $f$ has a continuous first derivative $f' : \mathcal{U} \to H_b' \cong H_b$, a continuous second derivative, $f'' : \mathcal{U} \to \mathcal{L}(H_b, H_b') \cong \mathcal{L}(H_b, H_b)$, and a third derivative $f''' : \mathcal{U} \to \mathcal{L}(H_b, \mathcal{L}(H_b, H_b)) \cong \mathcal{L}(H_b, \mathcal{L}(H_b, H_b))$ along the subspace $H$; and there exists a constant $K_f > 0$ such that*

$$\|f'(x)\| \leqslant K_f\ , \quad \|f''(x)\|_{\mathcal{L}(H_b,H_b)} \leqslant K_f\ , \quad \|f'''(x)\|_{\mathcal{L}(H_b,\mathcal{L}(H_b,H_b))} \leqslant K_f\ ,\ x \in \mathcal{U}\ ,$$

*where $\|\cdot\|_{\mathcal{L}(H_b,H_b)}$ and $\|\cdot\|_{\mathcal{L}(H_b,\mathcal{L}(H_b,H_b))}$ are the operator norms in the corresponding spaces.*
Assumption 3. *There exists $\varepsilon_b > 0$ such that the mapping $g_f$ can be extended to a one-to-one mapping*

$$\varphi\ : \mathcal{U} \times (-\varepsilon_b, \varepsilon_b) \to E\ , \quad (x,t) \mapsto g_f(x) + tn^G(x),$$

*and the inverse mapping $\psi = \varphi^{-1}$ defined on $G^{\varepsilon_b}$ is continuous.*

A measure on a topological space $X$ is understood to be a $\sigma$-additive function defined on a Borel $\sigma$-algebra $\mathfrak{B}_X$ of Borel subsets of the space $X$.

The differentiability of measures means $\tau_s$-differentiability [9].

Let $\nu$ be a non-negative Radon measure defined on $E$, two times differentiable along the space $H$, $\nu_G$ be a surface measure on $G$ defined in [35].

According to [35], on $\mathfrak{B}_G$, the function $\nu_G$ is given as follows:

$$\nu_G(Q) = \int\limits_Q \frac{\nu^d(dx)}{(n^G(P_b x), b))}\ ,$$

where $\nu^d$ is a measure on $G$ given by $\nu^d(Q) = d_b\nu(\bigcup_{s\leqslant 0}\{Q + sb\}),\ Q \in \mathfrak{B}_G$ . In this case, $\nu_G$ is a (non-negative) measure on $G$ [35].

Let $B \subset G$ be a subset. By Assumption 3, for every $\varepsilon \leqslant \varepsilon_b$, the set $B^\varepsilon$ is open if the set $B$ is open, and the set $B^\varepsilon$ is Borel if the set $B$ is Borel.

Let $\nu_T$ be the projection of $\nu$ onto $T_b$ along $R_b$. We say that a Borel subset $B \subset G$ possesses the property $(*)$ if $\nu_T(\partial P_b B) = 0$.

Theorem 1. *Let $B \subset G$ be an open subset which is contained in $G$ together with its closure and possesses the property $(*)$. Then,*

$$\nu_G(B) = \lim_{\varepsilon\to 0} \frac{\nu(B^\varepsilon)}{2\varepsilon}\ . \qquad (1)$$

*Outline of proof.* On $\mathcal{U} \times \mathbb{R}$, we define the function

$$f(x,\varepsilon) = f(x) + \frac{\varepsilon}{(n^G(x), b)} \ .$$

Let $B \subset G$ be an open set which possesses the property $(*)$. Define $f_\varepsilon(x) = f(x,\varepsilon)$, $B_{f,f_\varepsilon} \overset{\text{def}}{=} \{x + tb \ , \ x \in P_b B \ , \ f(x) \leqslant t < f_\varepsilon(x)\}$. By the results of Uglanov's paper [35],

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \nu(B_{f,f_\varepsilon}) = \int\limits_B (n^G(x), b) \frac{\partial f_\varepsilon}{\partial \varepsilon}(x,0) \nu_G(dx) = \nu_G(B) \ .$$

We prove that $\nu(B_\varepsilon \bigtriangleup B_{f,f_\varepsilon}) = o(\varepsilon)$, which will imply the formula (1). □

THEOREM 2. *Let the family of functions* $f_B(\varepsilon) = \dfrac{\nu^\varepsilon(B)}{\nu_G(B)}$, *where* $B \in \mathfrak{B}(G)$, *be uniformly bounded on the interval* $(0, \varepsilon_b]$. *Then, the formula* (1) *holds for every Borel subset of the surface* $G$.

*Outline of proof.* The proof of the theorem follows from the fact that the measures $\nu_G$ and $\nu$ possess the Radon property. □

## Chapter 2

In this chapter, we prove the Stokes formula for differential forms of a finite codegree and for submanifolds of a finite codimension in a locally convex space.

Let $\Xi_n$ $(n \in N)$ denote a vector space of all differentiable along the subspace $H$ differential forms of degree $n$ such that their differentials are continuous, and let, in addition to this, the ranges of values of differential forms from $\Xi_n$ and their differentials be bounded. By $S_n$, we denote the space of all differential forms of codegree $n$ differentiable along the subspace $H$ which are Radon measures and assume that the differential forms from $S_n$ and their differentials are measures of a bounded variation. We denote by $\bar{S}_n$ and $\bar{\Xi}_n$ $(n \in \mathbb{N})$ pseudo-topological vector spaces of all linear continuous functionals on $\Xi_n$ and $S_n$, respectively. We assume that $\bar{S}_n$ and $\bar{\Xi}_n$ contain $S_n$ and $\Xi_n$ as dense subsets. Let $V$ be a domain in the space $E$ such that its boundary $\partial V$ can be covered by a union of a finite number of surfaces (by a surface, we understand the object defined in the previous paragraph) $\mathcal{U}_i$ of codimension 1. Assume that the indicator $\mathbb{I}_V$ of the set $V$ is an element of the space $\bar{\Xi}_0$.

THEOREM 3. *Let* $\nu \in S_0$. *Then,* $(d\,\mathbb{I}_V) \cdot \nu \ (\in \bar{S}_1)$ *is a Radon measure on* $E$ *concentrated on* $\partial V$ *and taking values in* $H$, *and such that the following equality holds (for elements from* $\bar{S}_1$*)*

$$(d\,\mathbb{I}_V) \cdot \nu = -n^{\partial V} \cdot \nu^{\partial V} \ , \tag{2}$$

*where $\nu^{\partial V}$ denotes a surface measure $\partial V$ (in terms of the definition of Uglanov's paper [35]) generated by the measure $\nu$.*

Definition 3. *The integral from a differential form $\omega \in S_1$ over the surface on $\partial V$, we define in the following way:*

$$\int_{\partial V} \omega = \int_{\partial V} (n^{\partial V}, \omega^{\partial V}(dx)) \ ,$$

*where $\omega^{\partial V} \in S_1$ is such that its every component $\omega_p^{\partial V}$ is the surface measure generated by the measure $\omega_p$.*

Theorem 4 (The Stokes formula). *For $\omega \in S_1$, the formula*

$$\int_{\partial V} \omega = \int_V d\omega$$

*holds.*

*Outline of proof.* The Stokes formula can be proved applying Theorem 3. □

## Chapter 3

In this chapter, we investigate surface measures on manifolds of an infinite codimension. Here, the manifolds are totalities of continuous functions defined on a square and taking values in a compact Riemannian submanifold of a Euclidean space; the ambient spaces consist of continuous functions defined on the same square and taking values in the same Euclidean space. It is assumed, that in the ambient Euclidean space, the measure generated by a so called Brownian sheet, is given. One can consider the obtained result as a construction of a Brownian sheet with values in a compact Riemannian manifold. This result significantly generalizes an analogous result of P. Malliavin for Lie groups [18]. In this chapter, we also prove an analog of Chernoff's theorem for evolutional families of operators; using this analog, we construct a time inhomogeneous process on a compact Riemannian manifold.

Let $A_t$ be generators of strongly continuous semigroups on a Banach space $E$, such that the space $F = \cap_t D(A_t)$ is dense in $E$. Assume that for all $x \in F$, $\sup_{S \leqslant t \leqslant T} \|A_t x\|_E < \infty$. Introduce the norm in the space $F$: $\|x\|_F = \|x\|_E + \sup_t \|A_t x\|_E$. $F$ is a Banach space relative to this norm and all the operators $A_t : F \to E$ are continuous.

Consider a nonautonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = A_t u(t) \\ u(s) = x \ , \end{cases} \tag{3}$$

where $t, s \in [S, T]$, $s \leqslant t$. Further, we will assume that the Cauchy problem (3) is well-posed [13] (there exist several sufficient conditions of well-posedness of the Cauchy problem (3); references to the corresponding literature, one can find for example in [13]). According to [13], in this case, for the Cauchy problem (3), there exists a strongly continuous evolution family $U(t, s)$, $s, t \in [S, T], s \leqslant t$, solving the Cauchy problem.

Consider another nonautonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = -\bar{A}_t u(t) \\ u(r) = x \ , \end{cases} \tag{4}$$

for $t \leqslant r$ and $\bar{A}_t = A_{S+T-t}$.

LEMMA 1. *The Cauchy problem (4) is well-posed if and only if the Cauchy problem (3) is well-posed. In addition to this, the evolution family* $U(t, r)$*,* $t \leqslant r$*, solving the Cauchy problem (4) satisfies the following identity* $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$ *which holds for all* $t_1 \leqslant t_2 \leqslant t_3$*.*

THEOREM 5. *Let* $A_t$ *be generators of strongly continuous semigroups,* $B$ *be another generator of a strongly continuous semigroup. Assume that the following assumptions are fulfilled:*

1) *the space* $G = F \cap D(B)$ *is dense in* $E$*;*
2) *for every* $x \in F$*, the functions* $[S, T] \to E$ *,* $t \mapsto A_t x$*, are continuous;*
3) *the operator* $B$ *commutes with every operator* $A_t$*;*
4) $\{A_t\}$ *is a stable system of generators [34];*
5) *the Cauchy problem (4) is well-posed on* $F$*; let* $U(t, r)$ *be the evolution family solving the Cauchy problem (4);*
6) *there exists a dense in* $E$ *subset* $D \subset G$ *such that for all* $t$ *and* $r$ *holds:* $U(t, r)D \subset D$*,* $T_r(t)F \subset D$*;*
7) *for every fixed* $r$ *and* $x \in D$*, the families of functions* $s \mapsto A_t U(s, r)\, x$*,* $s \mapsto BU(s, r)\, x$ *are uniformly continuous;*
8) *for every* $y \in F$*, the family of functions* $s \mapsto A_{t+s} T_t(s)\, y$ *is uniformly continuous at the point* $s = 0$*.*

*Let* $Q_{t_1,t_2}$*,* $t_1, t_2 > 0$*, be a two-parameter family of linear contractions on* $E$ *such that* $\frac{Q_{\tau,\tau+\Delta\tau} - I}{\Delta\tau}\, e^{aB} x \to A_\tau\, e^{aB} x$*, as* $\Delta\tau \to 0$*, for all* $x \in D$*,* $a > 0$*, and uniformly in* $\tau$*. Let* $S \leqslant s < t \leqslant T$ *and* $\{s = t_0 < t_1 < \ldots < t_n = t\}$ *be a partition of the* *interval* $[s,t]$ *such that* $\max \Delta t_j \to 0$ *as* $n \to \infty$*, where* $\Delta t_j = t_{j+1} - t_j$ *. Then, for all* $x \in E$*,*

$$Q_{t_0,t_1} \dots Q_{t_{n-1},t_n} x \to U(s,t)\, x$$

*as* $n \to \infty$*.*

Let $M$ be a compact Riemannian manifold of dimension $d$ without boundary, isometrically embedded into $\mathbb{R}^m$. Let $\mathcal{P}_1 = \{t_0 = 0 < t_1 < \dots < t_n = 1\}$ be a partition of the interval $[0,1]$, $\varphi : [0,1] \to M - x$ be a differentiable function such that $\varphi(0) = 0$. If $E$ is a locally convex space, then $E^t$ denotes $C([0,t],E)$, the space of all continuous functions on $[0,t]$ with values in $E$. Every $\omega \in E^1$ can be identified with a sequence of $n$ elements $(\omega_1, \omega_2, \dots, \omega_n) \in E^{t_1} \times E^{t_2-t_1} \times \dots \times E^{t_n-t_{n-1}}$, where $\omega_j$ is defined on the interval $[0, t_j - t_{j-1}]$ by the formula $\omega_j(t) = \omega(t_{j-1}+t)$. Define the function $\varphi_{t_{i-1}t_i}$ on the interval $[0, t_i - t_{i-1}]$ by the formula

$$\varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1}+t) - \varphi(t_{i-1})\ .$$

Further, consider a process $(\mathbf{W}^z_{\psi,s})_t = \psi(t) + B^s_t$, where $\psi : [0,1] \to \mathbb{R}^m$ is a continuous function satisfying the condition $\psi(0) = 0$, and $B^s_t$ is a multiple Brownian motion with the parameter $s$ starting at the point $z$.

Let $\mathbb{W}^z_{\psi,s}$ denote the distribution of the process $(\mathbf{W}^z_{\psi,s})_t$, $\mathbb{E}_{z,\psi,s}$ denote the expectation relative this distribution.

In the following construction of a Brownian sheet on the manifold, we prove the existence of the limit

$$\int_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{z,\psi,s}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^z_{\psi,s})_t \in U_\varepsilon(M)\}}\}}{\mathbb{W}^z_{\psi,s}\{(\mathbf{W}^z_{\psi,s})_t \in U_\varepsilon(M)\}}$$

relative to the family of bounded continuous cylinder functions. This limit defines a measure $\mathbb{W}^z_{M,\psi,s,t}$. Using this measure, we define the measure $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ in the following way:

$$\int\limits_{C([0,1],\mathbb{R}^m)} h(\omega)\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,t_1],\mathbb{R}^m)} \mathbb{W}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int\limits_{C([0,t_2-t_1],\mathbb{R}^m)} \mathbb{W}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2) \dots \int\limits_{C([0,t_n-t_{n-1}],\mathbb{R}^m)} \mathbb{W}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) h(\omega_1,\omega_2,\dots,\omega_n). \tag{5}$$

THEOREM 6. *Let* $\varphi$ *be a differentiable function. Then, as the mesh of* $\mathcal{P}_1$ *tends to zero, the sequence of measures* $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ *converges weakly relative to the family of bounded continuous cylinder functions.*

*Outline of proof.* Chernoff's theorem for evolution families has to be applied to the family of generators $A_t f = (\varphi'(t), \nabla_M f)_{\mathbb{R}^m} - \frac{s}{2}\Delta_M$ . □

PROPOSITION 1. *Let $\imath$ be an isometric embedding of the manifold $M$ into $\mathbb{R}^m$, $g \in C^2(M)$. Then,*

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz) = g(y) + \frac{t}{8} g(y)\big(c(y) - scal(y)\big) - \frac{t}{2}\Delta_M g(y) + tR(t,y),$$

*where $|R(t,y)| < Kt^{1/2}$, $K$ is a constant which does not depend on $y$, $scal(y)$ is the scalar curvature at the point $y$; the function $c(y)$ has the form: $c(y) = \sum_{k,l}\sum_{\alpha}\Big(\frac{\partial^2 i^\alpha}{\partial x^k \partial x^l}\Big)^2(0)$, where $x^k$ are normal coordinates in a neighborhood $U_y$ of the point $y$ provided by a homeomorphism of the neighborhood $U_y$ onto a neighborhood of zero $U$ in $\mathbb{R}^d$; $i$ is the mapping $\imath$ in coordinates $x^k$. Independently of local coordinates, $c(y)$ can be written as follows: $c(y) = -\frac{1}{2}\Delta_M \Delta_M \left.|y - \cdot|^2\right|_y - \frac{1}{3}\, scal(y)$, and, hence, $c(y)$ depends only on the embedding $\imath$.*

*Outline of proof.* A more precise asymptotic, than the asymptotic from the paper of Smolyanov, Weizsäcker and Wittich [25], is obtained. The idea of the proof is the same. □

COROLLARY 1. *Let $g \in C^2(M)$. Then, the following asymptotic holds:*

$$\int_M g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz) \Big/ \int_M e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz) = g(y) - \frac{t}{2}\Delta_M g(y) + tR_1(t,y),$$

*where $|R_1(t,y)| < K_1 t^{1/2}$, $K_1$ is a constant which does not depend on $y$.*

PROPOSITION 2. *Let $\imath$ be an isometric embedding of the manifold $M$ into $\mathbb{R}^m$, $g \in C^2(M)$, $y \in M$, $0 < t < t_1 < 1$, $u_1$ and $u_2$ are such that $|u_2 - u_1| < t_1^\alpha$, $\alpha > 0$. Further, let $\mathrm{Pr}_M$ be mapping of projection onto the manifold $M$ along the subspaces normal to the manifold which are defined in the proper neighborhood of the manifold, $(u_2 - u_1)^y_M$ and $(u_2 - u_1)^y_\perp$ are such that*

$$(u_2 - u_1)^y_M = \mathrm{Pr}_M(y + u_2 - u_1) - y, \quad (u_2 - u_1)^y_\perp = y + u_2 - u_1 - \mathrm{Pr}_M(y + u_2 - u_1).$$

*Then, the following asymptotic holds:*

$$\frac{\int_M g(z) e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)} = g(y + (u_2 - u_1)^y_M) - \frac{t}{2}\Delta_M g(y + (u_2 - u_1)^y_M) + g(y + (u_2 - u_1)^y_M)\mathcal{R}(y, u_2 - u_1) + tR_2(t, t_1, y, u_2 - u_1); \quad (6)$$

*for the rest terms, we have:* $|R_2(t,t_1,y,u_2-u_1)| < K_2 t_1^{\alpha}$ *($K_2$ is a constant),*

$$\mathcal{R}(y,u_2-u_1) = -\frac{1}{4}(\Delta_M \imath(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)^2 + \sum_{n=3}^{N}\sum k(n) \prod_{l_1+\dots+l_s=n} D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}\big(y+(u_2-u_1)^y_M\big), \tag{7}$$

*where* $D^{2,l_i}_M$ *are differential operators on* $M$ *having the following form* $D^{2,l_i}_M = (\wedge_{k=1}^{l_i}\nabla_M^{(i_k)}, \wedge_{k=1}^{l_i}\nabla_M^{(j_k)})$*; the operators are applied to a product of* $l_i$ *functions,* $i_k$ *and* $j_k$ *are numbers from* 1 *till* $l_i$*, these numbers have the meaning of the number of function in the product, on which the corresponding operator acts; the index* 2 *says that* $i_k$ *and* $j_k$ *take the same value exactly* 2 *times,* $k(n)$ *are rational functions; the second sum in the last term contains a finite number of items, the number* $N$ *is chosen so that* $t_1^{(N-1)\alpha} < t$.

Now, we pass to the main result of the third chapter. As earlier, let $M$ be a compact Riemannian manifold of dimension $d$ without boundary, isometrically embedded into $\mathbb{R}^m$. A Brownian sheet with values in $\mathbb{R}^m$ is understood as a family of $m$ independent Brownian sheets. Let $\mathbf{W}_{t,s}$ be an $m$-dimensional Brownian sheet. Consider $\mathbf{W}_{t,s}$ as a process taking values in the space $C([0,1],\mathbb{R}^m)$. Denote this process by $\mathbf{W}_t$. Introduce the following notations: if $E$ is a locally convex space (LCS), then $E^t$ denotes $C([0,t],E)$; if $y \in C([0,1],\mathbb{R}^m)$ is a continuous function, then $\mathbb{W}^y$ denotes the distribution of the process $\mathbf{W}^y_t = y + \mathbf{W}_t$, for $\psi \in C([0,1],\mathbb{R}^m)$, we define a process $(\mathbf{W}^y_\psi)_t = \psi(t) + \mathbf{W}^y_t$. Let $\tilde{\mathbb{W}}^y_\psi$ be the distribution of this process, $\mathbb{E}_{y,\psi}$ be the expectation relative to the measure $\tilde{\mathbb{W}}^y_\psi$. Further, $U_\varepsilon(M)$ denotes the $\varepsilon$-neighborhood of the manifold $M$. We will consider $\mathbf{W}^y_\psi$ for functions $y$ and $\psi$ satisfying the following conditions: $y(0) \in M, \psi(0) = 0$. We prove the existence of the following limit relative to the family of bounded continuous cylinder functions, where everywhere below, by a cylinder function $C([0,1]\times[0,1],\mathbb{R}^m) \to \mathbb{R}$, we understand a function $f$ for which we can find a finite number of points $\tau_1,\dots,\tau_n$, $\xi_1,\dots,\xi_k$ and a function $\tilde{f} : \mathbb{R}^{nk} \to \mathbb{R}$ such that $f(\omega) = \tilde{f}(\omega(\tau_1,\xi_1),\omega(\tau_1,\xi_2),\dots,\omega(\tau_n,\xi_k))$. This limit defines a measure $\tilde{\mathbb{W}}^y_{M,\psi,s,t}$:

$$\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\tilde{\mathbb{W}}^y_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0}\frac{\mathbb{E}_{y,\psi}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^y_\psi)_t(s)\in U_\varepsilon(M)\}}\}}{\tilde{\mathbb{W}}^y_\psi\{(\mathbf{W}^y_\psi)_t(s)\in U_\varepsilon(M)\}}. \tag{8}$$

Before we prove the existence of this limit, we consider again the discussed above process $(\mathbf{W}^z_{\psi,s})_t = \psi(t) + B^s_t$, where $\psi : [0,1] \to \mathbb{R}^m$ is a continuous function satisfying the condition $\psi(0) = 0$. The results obtained for this process will be applied in the next construction.

*Some results for the process* $(\mathbf{W}^z_{\psi,s})_t$. Let $\mathbb{W}^z_{\psi,s}$ denote the distribution of the process $(\mathbf{W}^z_{\psi,s})_t$, $\mathbb{E}_{z,\psi,s}$ denote the expectation relative this distribution.

LEMMA 2. *The limit*

$$\int_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{z,\psi,s}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^z_{\psi,s})_t\in U_\varepsilon(M)\}}\}}{\mathbb{W}^z_{\psi,s}\{(\mathbf{W}^z_{\psi,s})_t \in U_\varepsilon(M)\}},$$

*exists relative to the family of bounded continuous cylinder functions and defines a measure* $\mathbb{W}^z_{M,\psi,s,t}$.

The process corresponding to the measure $\mathbb{W}^z_{M,\psi,s,t}$, we denote by $\mathbf{W}^z_{M,\psi,s,t}$.

*Outline of Proof.* Find the function $\tilde f : \mathbb{R}^{k+1} \to \mathbb{R}$ and a finite number of points $\tau_1,\dots,\tau_k$ such that $f(\omega) = \tilde f(\omega(\tau_1),\dots,\omega(\tau_k),\omega(t))$. We have

$$\int\limits_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0}\frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}\,\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega:\omega(t)\in U_\varepsilon(M)\}}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t)))}\cdot\int_{\mathbb{R}^m}\mathbb{P}^{\mathbb{W}}(\tau_1,0,dx_1)\int_{\mathbb{R}^m}\mathbb{P}^{\mathbb{W}}(\tau_2-\tau_1,x_1,dx_2)\dots$$

$$\int_{U_\varepsilon(M-\psi(t)-z)}\mathbb{P}^{\mathbb{W}}(t-\tau_k,x_k,dx_{k+1})\tilde f(x_1+z+\psi(\tau_1),\dots,x_k+z+\psi(\tau_k),x_{k+1}+z+\psi(t)),$$

where $\mathbb{P}^{\mathbb{W}}(\tau,x,dz) = \frac{1}{(2\pi s\tau)^{\frac{m}{2}}}e^{-\frac{|z-x|^2}{2s\tau}}dz$. Since the function under the integral sign is bounded, then, by the Lebesgue theorem, it is sufficient to prove the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\int\limits_{(U_\varepsilon(M-\psi(t)-z))}\tilde f(x_1+z+\psi(\tau_1),\dots,x_{k+1}+z+\psi(t))\mathbb{P}^{\mathbb{W}}(t-\tau_k,x_k,dx_{k+1})}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t)))}$$

$$= \lim_{\varepsilon\to 0}\frac{\int\limits_{U_\varepsilon(M-\psi(t)-z-x_k)}\tilde f(x_1+z+\psi(\tau_1),\dots,x_{k+1}+x_k+z+\psi(t))\mathbb{P}^{\mathbb{W}}(t-\tau_k,0,dx_{k+1})}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t)))}.$$

Denote by $M_1$ the manifold $M-\psi(t)-z-x_k$, by $M_2$ the manifold $M-\psi(t)-z$. Further, let $\lambda_\varepsilon = \frac{1}{vol_{m-d}(\varepsilon)}\,l|_{U_\varepsilon(M_1)}$, $\mu_\varepsilon = \frac{1}{vol_{m-d}(\varepsilon)}\,l|_{U_\varepsilon(M_2)}$, where $l$ is the Lebesgue measure on $\mathbb{R}^m$. It is easy to see that the proof of the existence of this limit can be reduced to the proof of the existence of the limit $\lim_{\varepsilon\to 0}\frac{\int_{\mathbb{R}^m} g(x_{k+1})\,e^{-\frac{|x_{k+1}-x_k|^2}{2s(t-\tau_k)}}\,\lambda_\varepsilon(dx_{k+1})}{\int_{\mathbb{R}^m} e^{-\frac{|x_{k+1}|^2}{2st}}\,\mu_\varepsilon(dx_{k+1})}$, where $g:\mathbb{R}^m\to\mathbb{R}$ is another notation for the function $\tilde f$, where the dependence just on the last variable is expressed. It is easy to prove that as $\varepsilon\to 0$, the measures $\lambda_\varepsilon$ and $\mu_\varepsilon$ converge weakly to the surface measures on $M_1$ and $M_2$, respectively. □

LEMMA 3. *The limit* (8) *exists relative to the family of bounded continuous cylinder functions.*

*Outline of proof.* Let $\mathbb{P}^{\tilde{\mathbb{W}}}(t, y, \Gamma) = \tilde{\mathbb{W}}^y(\omega : \omega(t) \in \Gamma)$ be the transition probability of the measure $\tilde{\mathbb{W}}^y$, where $y \in C([0,1], \mathbb{R}^m)$. Further, let the function $\tilde{f} : C([0,s], \mathbb{R}^m)^{k+1} \to \mathbb{R}$ and the finite number of points $\tau_1, \tau_2, \dots, \tau_k$, be such that $f(\omega) = \tilde{f}(\omega(\tau_1), \omega(\tau_2), \dots, \omega(\tau_k), \omega(t))$. The symbol $\pi_s$ denotes the coordinate mapping. The proof can be obtained by using a formula from the book [15] on the page 204:

$$\int_{C([0,s],\mathbb{R}^m)^t} f(\omega)\tilde{\mathbb{W}}^0(d\omega) = \int_{C([0,s],\mathbb{R}^m)} \mathbb{P}^{\tilde{\mathbb{W}}}(\tau_1, 0, dw_1) \int_{C([0,s],\mathbb{R}^m)} \mathbb{P}^{\tilde{\mathbb{W}}}(\tau_2 - \tau_1, w_1, dw_2) \dots \int_{\pi_s^{-1}(U_\varepsilon(M - \psi(t) - y(s)))} \tilde{f}(w_1, \dots, w_{k+1}) \mathbb{P}^{\tilde{\mathbb{W}}}(t - \tau_k, w_k, dw_{k+1}),$$

and applying Lemma 2 to the measure in the last integral. □

*The construction of the stochastic field* $\mathbf{W}_M^x$. Let $f$ be a bounded continuous cylinder function on $C([0,s], \mathbb{R}^m)^1$, $\varphi : \mathbb{R} \to M$ be a function which is a trajectory of a Brownian motion on $M$ and such that $\varphi(0) = x$. As earlier, $\mathcal{P}_1 = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ is a partition of the interval $[0,1]$ and $E^1 \ni \omega = (\omega_1, \omega_2, \dots, \omega_n) \in E^{t_1} \times E^{t_2 - t_1} \times \dots \times E^{t_n - t_{n-1}}$, where $\omega_j(t) = \omega(t_{j-1} + t)$, $\varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1} + t) - \varphi(t_{i-1})$, $t \in [0, t_j - t_{j-1}]$. Define a measure $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ by the formula:

$$\int_{C([0,s],\mathbb{R}^m)^1} f(\omega)\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) = \int_{C([0,s],\mathbb{R}^m)^{t_1}} \tilde{\mathbb{W}}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int_{C([0,s],\mathbb{R}^m)^{t_2 - t_1}} \tilde{\mathbb{W}}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2)$$
$$\dots \int_{C([0,s],\mathbb{R}^m)^{t_n - t_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) f(\omega_1, \omega_2, \dots, \omega_n).$$

We can check immediately that $\omega_i(t_i - t_{i-1})(0) \in M$, and hence, the measure $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ is well defined. Further, let $\mathcal{P}_2 = \{0 = s_0 < s_1 < \dots < s_k = 1\}$ be a partition of the interval $[0,1]$. Now we consider $s$ as the time parameter. Instead of the symbol $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$, we will use the symbol $\tilde{\mathbb{W}}^\varphi_{M,s,\mathcal{P}_1}$. Define the measure $\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ by the formula:

$$\int_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega) = \int_{C([0,1],\mathbb{R}^m)^{s_1}} \tilde{\mathbb{W}}^x_{M,s_1,\mathcal{P}_1}(d\omega_1) \int_{C([0,1],\mathbb{R}^m)^{s_2-s_1}} \tilde{\mathbb{W}}^{\omega_1(s_1)}_{M,s_2-s_1,\mathcal{P}_1}(d\omega_2) \dots$$
$$\int_{C([0,1],\mathbb{R}^m)^{s_n - s_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(s_{n-1}-s_{n-2})}_{M,s_n-s_{n-1},\mathcal{P}_1}(d\omega_n) f(\omega_1, \dots, \omega_n).$$

THEOREM 7 (About a Brownian sheet). *For every $x \in M$, as $|\mathcal{P}_1| \to 0$ and $|\mathcal{P}_2| \to 0$, $\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ converges weakly relative to the family of bounded continuous cylinder functions to a measure $\mathbb{W}^x_M$. The measure $\mathbb{W}^x_M$ has the following properties*

(i) *considered as the distribution of a* $C([0,1], M)$*-valued process, $\mathbb{W}^x_M$ possesses a transition probability at time t which is the distribution of a Brownian motion on $M$ with variance $t$ starting at the point $x$;*

(ii) *considered as the distribution of an $M$-valued two-parameter stochastic process, $\mathbb{W}^x_M$ possesses a transition probability at time $(s,t)$ which is the heat kernel measure on $M$ at time $st$, i.e., if $f \in \mathrm{C}(M)$, then*

$$\mathbb{W}^x_M \circ \pi^{-1}_{s,t}\,[f] = \left(e^{-\frac{st}{2}\Delta_M} f\right)(x),$$

*where $\pi_{s,t}$ is the mapping $\mathrm{C}([0,1]\times[0,1],M) \to M$, $\omega \mapsto \omega(s,t)$, $\Delta_M$ is the Laplace–Beltrami operator on $M$, $e^{-\frac{\tau}{2}\Delta_M}$ is the heat semigroup.*

*Outline of proof.* Let the measure $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ be defined exactly as in the case of a differentiable function $\varphi$, i.e., by the formula (5). First, let us assume that there exists a function $\tilde{f} : C([0,1],\mathbb{R}^m) \to \mathbb{R}$ such that $f(\omega) = \tilde{f}(\omega(t))$. Then,

$$\int\limits_{C([0,s],\mathbb{R}^m)^1} f(\omega)\,\tilde{\mathbb{W}}^{\varphi}_{M,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{f}(w)\,\tilde{\mathbb{W}}^{\varphi}_{M,s,\mathcal{P}_1} \circ \pi_s^{-1}(dw) = \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{f}(w)\,\mathbb{W}^{\varphi}_{M,s,\mathcal{P}_1}(dw),$$

From this, it follows:

$$\int\limits_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega) = \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^x_{M,s_1,\mathcal{P}_1}(dw_1) \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_1}_{M,s_2-s_1,\mathcal{P}_1}(dw_2)\ldots$$
$$\int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_{n-2}}_{M,s_{n-1}-s_{n-2},\mathcal{P}_1}(dw_{n-1}) \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_{n-1}}_{M,s_n-s_{n-1},\mathcal{P}_1}(dw_n)\tilde{f}(w_n).$$

Consider an integral of the form $\int_{C([0,t],\mathbb{R}^m)} g(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega)$, where the function $g \in C([0,t],\mathbb{R}^m)$ can be presented in the form $g(\omega) = \tilde{g}(\omega(t))$, $\tilde{g} \in C(\mathbb{R})$. After simple calculations, we get

$$\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}(\omega)\,\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega : \omega(t) \in U_\varepsilon(M)\}}$$
$$= \frac{\int\limits_M e^{-\frac{|x_1-z-\psi(t)|^2}{2ts}}\tilde{g}(x_1)\,\lambda_M(dx_1)}{\int\limits_M e^{-\frac{|x_1-z-\psi(t)|^2}{2ts}}\lambda_M(dx_1)}.$$

First, assume that the function $f$ is such that there exist a function $p : \mathbb{R}^m \to \mathbb{R}$ and numbers $t,s \in [0,1]$ such that $f(\omega) = p(\omega(t,s))$. The integral

$\int_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)$ has the form:

$$\frac{\int_M e^{-\frac{|x_1-x|^2}{2\Delta s_1\Delta t_1}}dx_1}{\int_M e^{-\frac{|\bar{x}_1-x|^2}{2\Delta s_1\Delta t_1}}d\bar{x}_1}\cdots\frac{\int_M e^{-\frac{|x_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}}dx_{n-1}}{\int_M e^{-\frac{|\bar{x}_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}}d\bar{x}_{n-1}}\frac{\int_M e^{-\frac{|x_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}}dx_n}{\int_M e^{-\frac{|\bar{x}_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}}d\bar{x}_n}$$
$$\frac{\int_M e^{-\frac{|y_1-x_1|^2}{2\Delta s_2\Delta t_1}}dy_1}{\int_M e^{-\frac{|\bar{y}_1-x_1|^2}{2\Delta s_2\Delta t_1}}d\bar{y}_1}\cdots\frac{\int_M e^{-\frac{|y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}}dy_{n-1}}{\int_M e^{-\frac{|\bar{y}_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}}d\bar{y}_{n-1}}\frac{\int_M e^{-\frac{|y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}}dy_n}{\int_M e^{-\frac{|\bar{y}_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}}d\bar{y}_n}$$
$$\cdots$$
$$\frac{\int_M e^{-\frac{|u_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}}du_1}{\int_M e^{-\frac{|\bar{u}_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}}d\bar{u}_1}\cdots\frac{\int_M e^{-\frac{|u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}}du_{n-1}}{\int_M e^{-\frac{|\bar{u}_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}}d\bar{u}_{n-1}}\frac{\int_M e^{-\frac{|u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}}du_n}{\int_M e^{-\frac{|\bar{u}_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}}d\bar{u}_n}$$
$$\frac{\int_M e^{-\frac{|v_1-u_1|^2}{2\Delta s_k\Delta t_1}}dv_1}{\int_M e^{-\frac{|\bar{v}_1-u_1|^2}{2\Delta s_k\Delta t_1}}d\bar{v}_1}\cdots\frac{\int_M e^{-\frac{|v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}}dv_{n-1}}{\int_M e^{-\frac{|\bar{v}_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}}d\bar{v}_{n-1}}\frac{\int_M e^{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}p(v_n)dv_n}{\int_M e^{-\frac{|\bar{v}_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}d\bar{v}_n},$$

where $\Delta t_i = t_i - t_{i-1}$, $\Delta s_j = s_j - s_{j-1}$, for simplicity of notations, instead of $\lambda_M(dz)$, we use $dz$. We also assumed here that $t_n = t$, $s_k = s$. Denote this integral by $I(\mathcal{P}_1, \mathcal{P}_2, p)$.

LEMMA 4. *$I(\mathcal{P}_1, \mathcal{P}_2, p)$ converges to $e^{-\frac{st}{2}\Delta_M}p$, as the meshes $|\mathcal{P}_1|$ and $|\mathcal{P}_2|$ tend to zero.*

*Outline of proof.* We choose $0 < \alpha < \tilde{\alpha} < \frac{1}{2}$, $\Delta\alpha = \tilde{\alpha} - \alpha$. In the integral of the form $\frac{\int_M e^{-\frac{|x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}dx_i}{\int_M e^{-\frac{|\bar{x}_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}d\bar{x}_i}$ at the $i$th place in the first line, we replace the integration over the whole manifold with the integration over a neighborhood of the point $x_{i-1}$ of radius $|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}}$. Here, we get a rest term not exceeding $\frac{C}{(2\pi\Delta s_1\Delta t_i)^{\frac{d}{2}}}e^{-\frac{1}{(\Delta s_1\Delta t_i)^{1-2\tilde{\alpha}}}}$, $C$ is a constant. For the $i$th integral of the second line, we have $|x_i - x_{i-1}| < |\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}}$. Further, we choose a neighborhood of the point $y_{i-1}$ on $M$ which contains only points $y_i$ satisfying the inequality $|y_i - y_{i-1}| < 2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}}$ holds. Again, we have a rest term not exceeding $\frac{C}{(2\pi\Delta s_2\Delta t_i)^{\frac{d}{2}}}e^{-\frac{C_1}{(\Delta s_2\Delta t_i)^{1-2\tilde{\alpha}}}}$, $C$, $C_1$ are constants. We continue fixing neighborhoods in the above way. Finally, we will consider the $i$th integral of the last line for points $u_i$ and $u_{i-1}$ satisfying $|u_i - u_{i-1}| < (K-1)^{\Delta\alpha}|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}} < |\mathcal{P}_2|^{\alpha}(\Delta t_i)^{\alpha}$, where the last inequality holds for sufficiently small meshes of $|\mathcal{P}_1|$, where $(K|\mathcal{P}_2|\Delta t_i)^{\Delta\alpha} < 1$. Further, starting from the last integral of the last line, we apply the asymptotic of Proposition 2. The asymptotic is being applied to each integral one after another. Terms containing expressions of the form $u_i - u_{i-1}$

disappear either after applying the integrals of the first line, or earlier. Thus, we get an asymptotic decomposition of the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$. We compare this decomposition with the decomposition for $e^{-\frac{st}{2}\Delta_M} = e^{-\frac{\Delta s_1 \Delta t_1}{2}\Delta_M} \dots e^{-\frac{\Delta s_k \Delta t_n}{2}\Delta_M}$, where we apply the asymptotic $e^{-\frac{\Delta s_i \Delta t_j}{2}\Delta_M} = 1 - \frac{\Delta s_i \Delta t_j}{2}\Delta_M + O((\Delta s_i \Delta t_j)^{\frac{3}{2}})$ to each of these exponents. Up to terms converging to zero as $|\mathcal{P}_1| \to 0$, $|\mathcal{P}_2| \to 0$, the decompositions will coincide. □

For a function $f$ depending on $\omega$ in several points, say, in the points $\xi_i \in [0, s]$ and $\tau_j \in [0, t]$, the form of the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$ will be the same, and the convergence can be proved analogously. It is assumed that the function $p = (x_{11}, \dots x_{ij}, \dots x_{lr})$ satisfies the equality $f(\omega) = p(\omega_{11}(\xi_1, \tau_1), \dots, \omega_{kl}(\xi_l, \tau_r))$, where $\omega_{ij}$ is defined on $[0, \xi_i - \xi_{i-1}] \times [0, \tau_j - \tau_{j-1}]$ by $\omega_{ij}(s, t) = \omega(\xi_{i-1} + s, \tau_{j-1} + t)$. The integral converges to a product of operators of the form $e^{-\frac{\Delta \xi_i \Delta \tau_j}{2}\Delta_M}$, each of the operators is applied to the function which is obtained from the function $p$ by fixing all variables besides the variable at the position $ij$. □

Corollary 2. *Let $M$ be a compact Lie group. Then, $\mathbf{W}_M^x$ considered as a $C([0,1], M)$-valued process, coincides with the Brownian motion constructed by Malliavin in [18].*

*Outline of proof.* The proof follows from Theorem 18 and Theorem 2.15 from the paper [12] (Theorem 2.15 from the paper [12] is proved also in Lemma 3.3 in the paper [33]). □

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# Chapter 1

# Approximation of surface measures on surfaces of a finite codimension in a locally convex space

## 1.1 Definitions and auxiliary results

All the vector spaces below are considered over the field of real numbers. All the topological spaces are Hausdorff.

Let $E$ be a locally convex space (LCS), $E'$ its dual space, $H$ a vector subspace of the space $E$ endowed with the structure of a Hilbert space with respect to the scalar product $(\cdot,\cdot)$ (we denote the corresponding norm by $\|\cdot\|$), $H'$ its dual space. Let $H$ be dense in $E$, and let the canonical embedding of $H$ into $E$ be continuous. Since, by the density of $H$ in $E$, the mapping $E' \to H'$ defined by the restriction of linear continuous on $E$ functionals to the space $H$ (the mapping adjoint to the embedding $H$ into $E$) is injective. Since $H'$ is identified with $H$ by the Riesz theorem, we have that $E'$ is embedded in $H$, and sometimes we will identify the elements of the spaces $E'$ and $H'$ with vectors of $H$ (wherein it is convenient and will not lead us to a contradiction). We denote by $R_a$ the one-dimensional subspace spanned on the unit vector $a \in H$. We fix the isometry between $R_a$ and the real axis, assuming that $ta \leftrightarrow t$ $(t \in \mathbb{R})$. In this sense (if the vector $a$ is fixed), we will identify $R_a$ and $\mathbb{R}$. A differentiability of functions is always understood to be Frechet differentiability [29].

Definition 4. *Let $D_F \subset E$ be an open set and $F : D_F \to \mathbb{R}$ be a continuous function having a derivative $F' : D_F \to H' \cong H$ along the subspace $H$. Let $S \stackrel{\text{def}}{=} \{x \in D_F \ : \ F(x) = 0\}$, and $F'(x) \neq 0$ for all $x \in S$. We call the vector $n^s = \frac{F'(x)}{\|F'(x)\|}$ (we identify the linear continuous on $H$ functional $F'(x)$ with an* *element of $H$) the normal vector to $S$ at the point $x$.*

Let the set $S$ be as in Definition 4, and $B \subset S$ be a set. For all $\varepsilon > 0$ we define the set

$$B^{\varepsilon} \stackrel{\text{def}}{=} \{x + tn^{s}(x) \ : \ x \in B \ , \ t \in (-\varepsilon, \varepsilon)\} \ .$$

DEFINITION 5. *We call the set $G$ a **surface**, if the following assumptions are fulfilled (1-3):*
ASSUMPTION 1. *The set $G$ is a graph of a continuous bounded function $f$, given on an open subset $\mathcal{U}$ of a closed space $T_b$ of codimension 1 of the space $E$ and taking values on the line $R_b = \{tb : t \in \mathbb{R}\}$, where $b \in H$ is a unit vector orthogonal in $H$ to the subspace $H_b \stackrel{\text{def}}{=} T_b \bigcap H$.*

REMARK 2. We denote by $P_b$ the projector operator on the subspace $T_b$ along the vector $b$.

REMARK 3. The subspace $H_b$ is dense in $T_b$ and is a closed subspace of codimension 1 of the space $H$.

Indeed, let $\mathcal{W} \subset T_b$ be an open (in $T_b$) set. Then, the set $\mathcal{W} + R_b$ is open in $E$. By the density of $H$ in $E$, there exists a point $x \in H \bigcap (\mathcal{W} + R_b)$. Hence $y = P_b x \in \mathcal{W} \bigcap H = \mathcal{W} \bigcap H_b$, and consequently $H_b$ is dense in $T_b$.

Further, the subspace $H_b$ is an orthogonal complement (in the space $H$) to $R_b$. Hence, it is closed in $H$ and is of codimension 1. From this, it also follows that $b \notin T_b$, and hence, $T_b + R_b = E$.

REMARK 4. Due to the existence of the isomorphism $T_b \times \mathbb{R}$ , $\quad (x,t) \mapsto x + tb$, we will identify the spaces $E$ and $T_b \times \mathbb{R}$ everywhere where it is convenient and cannot lead us to the contradiction. Here, we assume that the function $f$ takes values in $\mathbb{R}$. Similarly, we identify functions defined on $E$ and the corresponding functions defined on $T_b \times \mathbb{R}$ (in a natural way, using the above isomorphism).

ASSUMPTION 2. *The function $f$ has the continuous first-order derivative $f' \ : \mathcal{U} \to H_b' \cong H_b$, second-order derivative $f'' \ : \mathcal{U} \to \mathcal{L}(H_b, H_b') \cong \mathcal{L}(H_b, H_b)$ and the third-order derivative $f''' \ : \mathcal{U} \to \mathcal{L}(H_b, \mathcal{L}(H_b, H_b)) \cong \mathcal{L}(H_b, \mathcal{L}(H_b, H_b))$ and the third derivative $H$, and there exists a constant $K_f > 0$ such that*

$$\|f'(x)\| \leqslant K_f \ , \quad \|f''(x)\|_{\mathcal{L}(H_b,H_b)} \leqslant K_f \ , \quad \|f'''(x)\|_{\mathcal{L}(H_b,\mathcal{L}(H_b,H_b))} \leqslant K_f \ , \ x \in \mathcal{U},$$

*where $\|\cdot\|_{\mathcal{L}(H_b,H_b)}$ $\|\cdot\|_{\mathcal{L}(H_b,\mathcal{L}(H_b,H_b))}$ are the operator norms in the corresponding spaces.*

REMARK 5. One can present the set $G$ in the form

$$G = \{(x,t) \in \mathcal{U} \times \mathbb{R} \ : \ \Phi(x,t) = 0\} \ ,$$

where the function $\Phi$ is defined as follows:

$$\Phi \ : \mathcal{U} \times \mathbb{R} \to \mathbb{R} \ , \quad (x,t) \mapsto t - f(x) \ .$$

The function $\Phi$ is differentiable along the subspace $H$ (by the differentiability of $f$ along $H_b$), and everywhere, its derivative $\Phi' \ : \mathcal{U} \times \mathbb{R} \to H' \cong H$ is nonzero. Indeed, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $H$, and $e_1 = b$. Then,

$$\Phi'(x,t) = \sum_{n=1}^{\infty} (\Phi'(x,t), e_n) e_n = b - \sum_{n=2}^{\infty} (f'(x), e_n) e_n = b - f'(x) \ .$$

Hence, $\|\Phi'(x,t)\| = \sqrt{1 + \|f'(x)\|^2} > 0$. Thus, at each point $(x,t) \in G$ one can define the normal vector

$$n^G(x,t) = \frac{b - f'(x)}{\sqrt{1 + \|f'(x)\|^2}} \ , \tag{1.1}$$

and $(b, n^G(x,t)) = \dfrac{1}{\sqrt{1 + \|f'(x)\|^2}} > 0$. We could define the set $G$ with the help of function $-\Phi$ and then we would have $(b, n^G(x,t)) < 0$ for all $(x,t) \in G$.

Throughout the following (without loss of generality), we will assume that $(b, n^G(\omega)) > 0$ for all $\omega \in G$.

REMARK 6. Since there exists a bijection $g_f : \mathcal{U} \to G \ , \quad x \mapsto x + f(x)b$ of the domain $\mathcal{U}$ onto $G$, then up to the section 1.4 below, we will consider the domain $\mathcal{U}$ as the domain of the mapping $n^G$ (i.e., instead of $n^G(x,t)$, where $(x,t) \in G$, we write $n^G(x)$ assuming that $x \in \mathcal{U}$).

ASSUMPTION 3. *There exists $\varepsilon_b > 0$ such that the mapping $g_f$ can be extended to one-to-one mapping*

$$\varphi \ : \mathcal{U} \times (-\varepsilon_b, \varepsilon_b) \to E \ , \quad (x,t) \mapsto g_f(x) + t n^G(x),$$

*and the inverse mapping $\psi = \varphi^{-1}$ defined on $G^{\varepsilon_b}$, is continuous.*

LEMMA 5. *For all $x \in \mathcal{U}$, $(n^G)'(x) \in \mathcal{L}(H_b, H)$, $(n^G)''(x) \in \mathcal{L}(H_b, \mathcal{L}(H_b, H))$, and there exists a constant $K_n > 0$, such that*

$$\|(n^G)'(x)\|_{\mathcal{L}(H_b,H)} < K_n \ , \quad \|(n^G)''(x)\|_{\mathcal{L}(H_b,\mathcal{L}(H_b,H))} < K_n \ . \tag{1.2}$$

*Proof.* Differentiating the relation (1.1) along an arbitrary direction $h \in H_b$, we obtain

$$(n^G)'(x)h = \frac{(f'(x) - b)(f'(x), f''(x)h)}{(1 + \|f'(x)\|^2)^{\frac{3}{2}}} - \frac{f''(x)h}{(1 + \|f'(x)\|^2)^{\frac{1}{2}}} \ . \tag{1.3}$$

This and Assumption 2) of Definition 4 imply the inclusion $(n^G)'(x) \in \mathcal{L}(H_b, H)$, and relation (1.3) implies the estimate

$$(n^G)'(x)\| \leqslant (\|f'(x)\| + 1)\|f'(x)\|\|f''(x)\| + \|f''(x)\| < (K_f + 1)K_f^2 + K_f \ .$$

Differentiating (1.3) along an arbitrary vector $h_1 \in H_b$ and again taking into account Assumption 2) of Definition 4, we obtain the second estimate in (1.2).

□

LEMMA 6. *Let $X$ be a LCS, $X_1$ be its vector subspace equipped with the locally convex topology, and let the mapping $\Psi : E \supset D(\Psi) \to X$ have the derivative $\Psi' : D(\Psi) \to \mathcal{L}(H, X_1)$. Assume that the curve $\gamma : [0,1] \to D(\Psi)$ is such that at some point $t_0 \in [0,1]$ there exists the derivative $\dot\gamma(t_0)$ with respect to the topology of the space $X1$. Then, at the point $t_0$, there exists the derivative of the composite function $\Psi \circ \gamma$, and*

$$\frac{d}{dt}(\Psi \circ \gamma)(t)\bigg|_{t=t_0} = \Psi'(\gamma(t_0))\dot\gamma(t_0) \ .$$

REMARK 7. The fact that the derivative $\dot\gamma(t_0)$ exists with respect to the topology of the space $X1$ means that for all $t$ from some neighborhood of $t_0$, $\gamma(t) - \gamma(t_0) \in H$, and the limit $\lim\limits_{t \to t_0} \dfrac{\gamma(t) - \gamma(t_0)}{t - t_0}$ exists with respect to the topology of the space $X1$.

*Proof of Lemma 6.* Let $\omega \overset{\text{def}}{=} \gamma(t_0)$. We introduce the function

$$\Psi_\omega : H \supset D(\Psi_\omega) \to E \ , \quad h \mapsto \Psi(h + \omega)$$

and the curve

$$\gamma_0 : [0,1] \to H \ , \quad t \mapsto \gamma(t) - \gamma(t_0) \ .$$

Then $\Psi \circ \gamma = \Psi_\omega \circ \gamma_0$. But for the function $\Psi_\omega \circ \gamma_0$, we can apply the theorem on the differentiability of the composition at the point $t_0$. Taking into account that $\Psi'_\omega(0) = \Psi'(\omega)$ and $\dot\gamma_0(t_0) = \dot\gamma(t_0)$, we get

$$\frac{d}{dt}(\Psi \circ \gamma)(t)\bigg|_{t=t_0} = \frac{d}{dt}(\Psi_\omega \circ \gamma_0)(t)\bigg|_{t=t_0} = \Psi'_\omega(0)\dot\gamma_0(t_0) = \Psi'(\gamma(t_0))\dot\gamma(t_0) \ .$$

□

LEMMA 7. *The mapping $\varphi$ is continuous on $\mathcal{U} \times (-\varepsilon_b, \varepsilon_b)$ and has the first-order derivative $\varphi' : \mathcal{U} \times (-\varepsilon_b, \varepsilon_b) \to \mathcal{L}(H, E)$ and and the second-order derivative $\varphi'' : \mathcal{U} \times (-\varepsilon_b, \varepsilon_b) \to \mathcal{L}(H, \mathcal{L}(H, E))$ along the subspace $H$, and $\varphi'$ takes values in the space $\mathcal{L}(H, H)$, $\varphi''$, in the space $\mathcal{L}(H, \mathcal{L}(H, H))$, and there exists a constant $K_\varphi > 0$, such that for all $x \in \mathcal{U} \times (-\varepsilon_b, \varepsilon_b)$*

$$\|\varphi'(x)\|_{\mathcal{L}(H,H)} < K_\varphi \ , \quad \|\varphi''(x)\|_{\mathcal{L}(H,\mathcal{L}(H,H))} < K_\varphi \ .$$

*where $\|\cdot\|_{\mathcal{L}(H,H)}$ and $\|\cdot\|_{\mathcal{L}(H,\mathcal{L}(H,H))}$ are the operator norms.*

*Proof.* The mapping $\varphi$ is continuous by its definition (Assumption 3 of Definition 4), and by continuity of the mapping $n^G$, which is implied by (1.1).

Let us show its differentiability. Let $(x, t) \in \mathcal{U} \times (-\varepsilon_b, \varepsilon_b)$, $h \in H, \tau \in \mathbb{R}$.

$$\begin{aligned}\varphi(x + tb + \tau h) &= \varphi(x + \tau P_b h + (t + \tau(h, b))b) = \\ &= x + \tau P_b h + f(x + \tau P_b h)b + (t + \tau(b, h))n^G(x + \tau P_b h) \ .\end{aligned}$$

Taking the derivative of this expression with respect to $\tau$ at the point $\tau = 0$ and using Lemmas 6 and 5, we get

$$\begin{aligned}\varphi'(x, t)h = \left.\frac{d\varphi}{d\tau}\right|_{\tau=0} &= P_b h + (f'(x), P_b h)b + (b, h)n^G(x) + \\ &+ t(n^G)'(x)P_b h = (f'(x) - b, h)b + (b, h)n^G(x) + h + t(n^G)'(x)P_b h \ . \quad (1.4)\end{aligned}$$

By Lemma 5, $(n^G)'(x) \in \mathcal{L}(H_b, H)$, From this it follows that $\varphi'(x, t) \in \mathcal{L}(H, H)$. Differentiating equality (1.4) and again taking into account Lemma 5, we obtain that $\varphi''(x, t) \in \mathcal{L}(H, \mathcal{L}(H, H))$. The boundedness of the derivatives $\varphi'$ and $\varphi''$ follows from Assumption 2 of Definition 4 and Lemma 5. □

LEMMA 8. *There exists $\varepsilon_b' < \varepsilon_b$, such that the mapping $\psi$ has the first-order derivative $\psi' : G^{\varepsilon_b'} \to \mathcal{L}(H, H)$ and the second-order derivative $\psi'' : G^{\varepsilon_b'} \to \mathcal{L}(H, \mathcal{L}(H, H))$ along the subspace $H$, and there exists a constant $K_\psi$, such that for all $x \in G^{\varepsilon_b'}$*

$$\|\psi'(x)\|_{\mathcal{L}(H,H)} < K_\psi \ , \quad \|\psi''(x)\|_{\mathcal{L}(H,\mathcal{L}(H,H))} < K_\psi \ .$$

Below we will omit the indices in $\|\cdot\|_{\mathcal{L}(H,H)}$ and $\|\cdot\|_{\mathcal{L}(H,\mathcal{L}(H,H))}$.

*Proof.* For each $\omega \in \mathcal{U} \times (-\varepsilon_b, \varepsilon_b)$ we define the function

$$\varphi_\omega \ : \ (\mathcal{U} \times (-\varepsilon_b, \varepsilon_b) - \omega) \bigcap H \to H \ , \quad u \mapsto \varphi(\omega + u) - \varphi(\omega) \ .$$

Similarly, for each $\omega \in G^{\varepsilon_b}$ we define the function

$$\psi_\omega \ : G^{\varepsilon_b} \bigcap H \to H \ , \quad u \mapsto \psi(\omega + u) - \psi(\omega) \ .$$

Fix an arbitrary point $\omega_0 \in G^{\varepsilon_b}$. One can easily check that $\varphi_{\psi(\omega_0)}(\psi_{\omega_0}(u)) = u$. We denote $\psi(\omega_0)$ by $w_0$ and consider the equation $\varphi_{w_0}(v) = u$. We will show that for some $\varepsilon_b' < \varepsilon_b$ for $\omega_0 \in G^{\varepsilon_b'}$ all the conditions of the inverse function theorem are fulfilled for the function $\psi_{\omega_0}$ at the point $u = 0$ [23]. Indeed, for all $w \in \mathcal{U} \times (-\varepsilon_b, \varepsilon_b)$ $(\varphi_w)'(v) = \varphi'(w+v) \in \mathcal{L}(H,H)$, and $(\varphi_w)'$ is a continuous function on its domain, since it is a differentiable function on this domain. Further, we will find $\varepsilon_b' < \varepsilon_b$, such that for $w \in \mathcal{U} \times (-\varepsilon_b', \varepsilon_b')$ the operator $(\varphi_w)'(0)$ is invertible. Let first $w \in \mathcal{U}$. We present $w$ in the form $w = (x, 0)$, where $x \in \mathcal{U}$. By Lemma 1.4, $(\varphi_w)' = \varphi'(x,0) \in \mathcal{L}(H,H)$. We will prove that $\varphi'(x,0)$ is a bijection. From this, by the Banach inverse operator theorem, we will get that $(\varphi'(x,0))^{-1} \in \mathcal{L}(H,H)$.

Let $h \in H$. From the proof of Lemma 1.4 we have

$$\varphi'(x,0)h = (f'(x) - b, h)b + (b,h)n^G(x) + h = (K+I)h \ ,$$

where $K = (f'(x) - b, h)b + (b,h)n^G(x)$is a finite dimensional operator such that $\ker(K+I) = \{0\}$. Indeed, each vector $h \in \ker(K+I)$ can be presented in the form $h = -Kh$, and hence $h \in \operatorname{Im} K$. Therefore in the case of the linear independence of the vectors $b$ and $n^G(x)$, there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, such that $h = \alpha b + \beta n^G(x)$. Substituting this expression in the equation

$$(K+I)h = 0 \ , \tag{1.5}$$

and taking into account (1.1), we obtain the following system of equations relative to $\alpha$ and $\beta$:

$$\begin{cases} -\beta\sqrt{1 + \|f'(x)\|^2} = 0 \\ \alpha + \beta(1 + (b, n^G(x))) = 0 \ . \end{cases}$$

This system has obviously only the trivial solution, and hence $h = 0$. If the vectors $b$ and $n^G(x)$ are linear dependent, then $b = n^G(x)$. In this case the solution of equation (2.9) has the form $h = \alpha b$, where $\alpha \in \mathbb{R}$. Substituting this in (2.9), we get again $h = 0$. By Fredholm's theorem the equation $(K+I)h = h'$ is uniquely solvable for all $h' \in H$, i.e. $\varphi'(x,0)$ $(= K+I)$ is a bijection.

Thus, for all $w \in \mathcal{U}$ the operator $(\varphi_w)'(0) = \varphi'(w)$ is invertible, and, by the theorem on differentiation of inverse function [23], for all $\omega \in G$, $\psi'(\omega) = (\psi_\omega)'(0) = ((\varphi_w)'(0))^{-1} \in \mathcal{L}(H,H)$ $(w = \psi(\omega))$.

Further, we will show that $\psi'(\omega)$, as a function of $\omega$, is bounded on $G$ with respect to the norm of the space $\mathcal{L}(H,H)$. For $\omega \in G$ we will find the explicit form $\psi'(\omega)$. For this, we let $\omega = (x, f(x))$ and for all $h' \in H$ solve the following equation with respect to $h$

$$(f'(x) - b, h)b + (b,h)n^G(x) + h = h' \ . \tag{1.6}$$

First, we consider the case $b \neq n^G(x)$, which means the linear independence of the vectors $b$ and $n^G(x)$. Taking into account that $h = h' - (f'(x) - b, h)b -$

$(b,h)n^G(x)$, we will search $h$ in the form $h = h' + \alpha b + \beta n^G(x)$. Substituting this in equation (2.11), we get the following expressions for $\alpha$ and $\beta$:

$$\beta = \frac{(f'(x) - b, h')}{\sqrt{1 + \|f'(x)\|^2}} , \quad \alpha = -\beta - (b, h' + \beta n^G(x)) .$$

Using these equalities and taking into account the uniqueness of the solution, we get the following estimate

$$\|\psi'(\omega)h'\| = \|h' + \alpha b + \beta n^G(x)\| \leqslant \|h'\| + |\alpha| + |\beta| .$$

Further,

$$|\beta| \leqslant \frac{\|f'(x) - b\|}{\sqrt{1 + \|f'(x)\|^2}} \|h'\| = \|h'\| ;$$
$$|\alpha| \leqslant 2|\beta| + \|h'\| \leqslant 3\|h'\| .$$

Therefore $\|\psi'(\omega)\| \leqslant 5$ for all $\omega \in G$. Now let $b = n^G(x)$. We will search the solution of (2.11) in the form $h = h' + \alpha b$. Substituting this in equation (2.11), we get $\alpha = -(f'(x), h')$. From this we have $\|\psi'(\omega)\| \leqslant 1 + K_f$.

Further, rewrite the expression for $\varphi'(x,t)$, where $x \in \mathcal{U}$, $t \in (-\varepsilon_b, \varepsilon_b)$ (see the proof of Lemma 1.4)

$$\varphi'(x,t)h = (f'(x) - b, h)b + (b,h)n^G(x) + h + t(n^G)'(x)P_b h = A_0 h + \Delta A_t h ,$$

where $A_0 = \varphi'(x,0)$, $\Delta A_t = t(n^G)'(x)P_b$. Lemma 5 gives the estimate $\|\Delta A_t\| \leqslant \|(n^G)'(x)\| t \leqslant K_n t$, and hence, there exists $t_0 > 0$, such that for all $t \in (-t_0, t_0)$

$$\|\Delta A_t\| < \frac{1}{2K} , \tag{1.7}$$

where $K = \max\{5, K_f + 1\}$. Therefore, $\|\Delta A_t\| < \|A_0^{-1}\|^{-1}$. By the theorem on the invertibility of perturbed operators, for any $t \in (-t_0, t_0)$, the operator $(A_0 + \Delta A_t)^{-1} \in \mathcal{L}(H,H)$ exists. Choose $\varepsilon_b'$ be equal to $t_0$. Again applying the theorem on differentiability of the inverse function, we obtain that for all $\omega \in G^{\varepsilon_b'}$, $\psi'(\omega) = (\psi_\omega)'(0) = ((\varphi_w)'(0))^{-1} \in \mathcal{L}(H,H)$, where $w = \psi(\omega)$.

Further,

$$(A_0 + \Delta A_t)^{-1} = (I + A_0^{-1}\Delta A_t)^{-1} A_0^{-1} = A_0^{-1} + \left( \sum_{n=1}^{\infty} (-1)^n (A_0^{-1}\Delta A_t)^n \right) A_0^{-1} .$$

From this, by (1.7), we get

$$\|(A_0 + \Delta A_t)^{-1}\| \leqslant \|A_0^{-1}\| + \left( \sum_{n=1}^{\infty} \|A_0^{-1}\|^n \|\Delta A_t\|^n \right) \|A_0^{-1}\| \leqslant K \left( 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \right) = 2K .$$

In other words, for all $\omega \in G^{\varepsilon_b'}$, $\|\psi'(\omega)\| \leqslant 2K$.

Let us now show the differentiability of the function $(\psi_{\omega_0})'$ ($\omega_0$ has been fixed, $w_0 = \psi(\omega_0)$). Since the function $(\varphi_{w_0})'$ 0 is differentiable on $(\mathcal{U} \times (-\varepsilon_b', \varepsilon_b') - w_0) \bigcap H$, then it is continuous. The operator function $\mathcal{L}(H,H) \to \mathcal{L}(H,H)$ , $A \mapsto A^{-1}$ is also continuous at each point $A \neq 0$. Since $(\varphi_{w_0})'$, is nonzero on $(\mathcal{U} \times (-\varepsilon_b', \varepsilon_b') - w_0) \bigcap H$ (by invertibility), then $(\psi_{\omega_0})'$ is continuous on $(G^{\varepsilon_b'} - \omega_0) \bigcap H$ as a composition of continuous functions. Taking this into account, we get

$$\psi''(\omega_0)h = (\psi_{\omega_0})''(0)h = \lim_{t\to 0} \frac{(\psi_{\omega_0})'(th) - (\psi_{\omega_0})'(0)}{t} =$$
$$= -\lim_{t\to 0} \frac{(\psi_{\omega_0})'(th)((\varphi_{w_0})'(th) - (\varphi_{w_0})'(0))(\psi_{\omega_0})'(0)}{t} = -\psi_{\omega_0})'(0)(\varphi_{w_0})''h\psi_{\omega_0})'(0) =$$
$$= -\psi'(\omega_0)\varphi''(\psi(\omega_0))h\psi'(\omega_0), \quad h \in H \ .$$

From this, for all $\omega \in G^{\varepsilon_b'}$ we get the following estimate:

$$\|\psi''(\omega)\| \leqslant \|\psi'(\omega)\|^2 \|\varphi''(\omega)\| \leqslant K^2 K_\varphi \ .$$

□

Without loss of generality we will consider that $\varepsilon_b' = \varepsilon_b$.

The following theorem will be needed below.

THEOREM 8 (on the inverse function). *Suppose $D_F \subset E$ is a domain, and $F : D_F \to \mathbb{R}$ is a continuous function with the continuous derivative $F' : D_F \to H' \cong H$ along the subspace $H$. Let $S = \{\omega \in D_F \ : \ F(\omega) = 0\}$, and let the derivative $F'$ is nonzero at each point $\omega \in S$. Further, let $T_b$ be a closed subspace of codimension 1 of the space $E$, and let the unit vector $b \in j_H(E')$, which is orthogonal in $H$ to the subspace $H_b = T_b \bigcap H$, be such that $(n^s(\omega), b) > 0$ for all $\omega \in S$ (here $n^s(\omega) = \dfrac{F'(\omega)}{\|F'(\omega)\|}$, and, as before, we identify the space $E$ with the space $T_b \times \mathbb{R}$ by using the isomorphism $T_b \times \mathbb{R} \to E$ , $(x,t) \mapsto x + tb$). Then*

1) *there exists an open (in $T_b$) set $\mathcal{V} \subset T_b$ and a continuous function $g : \mathcal{V} \to \mathbb{R}$, such that $F(x, g(x)) = 0$ for all $x \in \mathcal{V}$*

2) *the derivative $g' : \mathcal{U} \to H' \cong H$ of $g$ along the subspace $H$ has the form $g'(x) = \dfrac{P_b n^s(x, g(x))}{(b, n^s(x, g(x)))}$;*

3) *if the function $F$ has all derivatives of order $k \geqslant 2$ along the subspace $H$, then the function $g$ also has all derivatives of order $k \geqslant 2$ along $H_b$;*

*4) moreover, if there exist constants $\lambda > 0$, $\sigma > 0$, and $C > 0$, such that*

$$\|F'(\omega)\| > \lambda, \quad (b, n^s(\omega)) > \sigma, \quad \|F''(\omega)\| < C \ ,$$

*for all $\omega \in S$, then there exists a constant $C_g$, such that $\|g''(x)\| < C_g$ for all $x \in \mathcal{V}$.*

*Proof.* Assertion 1 holds by the implicit function theorem [23]. Indeed, the continuous partial derivative $\frac{\partial}{\partial t}F(x,t)$ exists at each point $(x,t) \in D_F$; this derivative is different from zero at each point $(x,t) \in S$ because

$$\frac{\partial}{\partial t}F(x,t) = F'(x+tb)b = \|F'(x+tb)\|(n(x+tb),b) > \sigma\|F'(x+tb)\| \ .$$

Next, we fix an arbitrary point $\omega = (x_0, g(x_0)) \in S$, where $x_0 \in \mathcal{V}$, and define the functions:

$$F_\omega \ : \ (D_F - \omega)\bigcap H \to \mathbb{R} \ , \quad u \mapsto F(\omega + u) \ ;$$
$$g_{x_0} \ : \ (\mathcal{V} - x_0)\bigcap H_b \to \mathbb{R} \ , \quad x \mapsto g(x_0 + x) - g(x_0) \ .$$

Since $F(x, g(x)) = 0$ for $x \in \mathcal{V}$, then $F_\omega(x, g_{x_0}(x)) = 0$. The function $F_\omega$ satisfies all the assumptions of the theorem on the differentiability of the implicit function $g_{x_0}$ at the point $(0,0)$. Indeed, $\frac{\partial}{\partial t}F(x,t)$ is a continuous function $(D_F - \omega)\bigcap H \to \mathbb{R}$, since

$$\frac{\partial}{\partial t}F_\omega(x,t) = \frac{\partial}{\partial t}F(x + x_0, t + g(x_0)) = F'(x + x_0 + (t + f(x_0))b)b \ . \tag{1.8}$$

Further, $\frac{\partial}{\partial t}F_\omega(0,0) \neq 0$, since, by (1.8)

$$\frac{\partial}{\partial t}F_\omega(0,0) = \|F'(x_0 + g(x_0)b)\|(n^s(x_0), b) > 0 \ .$$

Finally, the partial derivative $\frac{\partial}{\partial x}F_\omega(0,0) = F'(x_0 + g(x_0)b)$ exists at the point $(0,0)$. Then, by the implicit function theorem, the function $g_{x_0}$ has the derivative

$$(g_{x_0})'(0) = -\frac{\partial}{\partial x}F_\omega(0,0)\Big/\frac{\partial}{\partial t}F_\omega(0,0)$$

at the point $x = 0$. Taking into account the relation $(g_{x_0})'(0) = g'(x_0)$ and the fact that $x_0 \in \mathcal{V}$ is an arbitrary point, we transform the expression in the right-hand side and obtain

$$g'(x)h = -\frac{F'(x+g(x)b)h}{F'(x+g(x)b)b} = -\frac{\|F'(x+g(x)b)\|(n^s(x),h)}{\|F'(x+g(x)b)\|(n^s(x),b)} = \frac{(n^s(x),h)}{(n^s(x),b)}$$
$$= \frac{(P_b n^s(x),h)}{(n^s(x),b)} \tag{1.9}$$

for all $\in \mathcal{V}$ and for all $h \in H_b$. Hence, as an element of the space $H_b$, $g'(x)$ can be written in the form $g'(x) = \dfrac{P_b n^s(x)}{(n^s(x), b)}$.

Assertion 3 immediately follows from (1.9), and for the second-order derivative of the function $g$ we obtain the expression:

$$g''(x)hh_1 = -\frac{F''(x+g(x)b)hh_1 + (g'(x)h_1)F''(x+g(x)b)hb}{F'(x+f(x)b)b} + \\ + \frac{(F'(x+g(x)b)h)F''(x+g(x)b)bh + (g'(x)h_1)F''(x+g(x)b)bb}{(F'(x+g(x)b)b)^2} ,$$

from which, using the relation $F'(x+g(x)b)b = \|F'(x+g(x)b)\|(n^s(x), b)$, we obtain Assertion 4. □

Remark 8. We could have defined the surface $G$ using the function $F$, defined on some open set $D_F \subset E$ and satisfying the assumptions of Theorem 8, as a set of points of the form $\{\omega \in D_F \ : \ F(\omega) = 0\}$, assuming that the third-order derivative of the function $F$ exists and is bounded on $D_F$ relative to the norm of the space $\mathcal{L}(H, \mathcal{L}(H, H))$. Then, by Theorem 8, Assumptions 1)–3) of Definition 5 would have been satisfied.

## 1.2 The surface layer theorems

A measure on a topological space $X$ is understood to be a $\sigma$-additive function defined on a Borel $\sigma$-algebra $\mathfrak{B}_X$ of Borel subsets of the space $X$.

The differentiability of measures means $\tau_s$-differentiability [9].

Let $\nu$ be a non-negative Radon measure, defined on $E$, two times differentiable along the space $H$; $\nu_{_G}$ be a surface measure on $G$ defined in [35].

According to [35], on $\mathfrak{B}_G$ the function $\nu_{_G}$ is given as follows:

$$\nu_{_G}(Q) = \int\limits_Q \frac{\nu^d(dx)}{(n^{_G}(P_b x), b))} , \tag{1.10}$$

where $\nu^d$ is a measure on $G$ given by $\nu^d(Q) = d_b \nu(\bigcup\limits_{s \leqslant 0} \{Q + sb\}),\ Q \in \mathfrak{B}_G$ . In this case $\nu_{_G}$ is a (non-negative) measure on $G$ [35].

Let $B \subset G$ be a subset. By Assumption 3, for every $\varepsilon \leqslant \varepsilon_b$, the set $B^\varepsilon$ is open if the set $B$ is open, and the set $B^\varepsilon$ is Borel if the set $B$ is Borel.

For $B \in \mathfrak{B}_G$ and for each $\varepsilon \leqslant \varepsilon_b$, we define the set

$$B_\varepsilon \stackrel{\text{def}}{=} \{x + tn^{_G}(x)\ ,\ x \in B, 0 \leqslant t < \varepsilon\} , \tag{1.11}$$

which is a Borel set, since the mapping $\psi$ is continuous.

Let $\nu_{_T}$ be the projection of $\nu$ onto $T_b$ along $R_b$. We say that a Borel subset $B \subset G$ possesses the property $(*)$ if $\nu_{_T}(\partial P_b B) = 0$.

THEOREM 9. *Let $B \subset G$ be an open subset which is contained in $G$ together with its closure, and possesses the property $(*)$. Then,*

$$\nu_{_G}(B) = \lim_{\varepsilon \to 0} \frac{\nu(B^\varepsilon)}{2\varepsilon} \ . \tag{1.12}$$

For the proof of the theorem we need the following lemmas (9–14).

LEMMA 9. *Suppose that $A \in \mathfrak{B}_{\mathcal{U}}$, $f_1$ , $f_2$ : $A \to R_b$ are functions, $f_1 \leqslant f_2$, and*

$$A_{f_1,f_2} \stackrel{\text{def}}{=} \{\omega \in E \ : \ P_b\omega \in A \ , f_1(P_b\omega) \leqslant \omega - P_b\omega \leqslant f_2(P_b\omega)\} \ .$$

*Suppose also that $Q \subset A_{f_1,f_2}$ is a Borel set. Then the following inequality holds:*

$$\nu(Q) \leqslant 4|d_b\nu|(A + R_b) \sup(f_2 - f_1) \ .$$

*Proof.* This lemma is proved [36] for a separable Banach space. In our case the proof does not change. □

We define the mapping $P_{_G}$ of "projecting on $G$ along the normal vector" as follows:

$$P_{_G} \ : \ G^{\varepsilon_b} \to G \ , \quad x + tn^{G}(x) \mapsto x \ .$$

It is easy to see that $P_{_G} = \varphi(\cdot, 0) \circ P_b \circ \psi$. Hence, the mapping $P_{_G}$ is continuous as a composition of continuous mappings.

We set $G_\varepsilon = \varphi(\mathcal{U} + \varepsilon b)$. By the symbol $L_{\varepsilon,x}$, we denote the set of all vectors tangent to $G_\varepsilon$ at the point $x + \varepsilon n^G(x)$, where $x \in \mathcal{U}$, such that for each of these vectors there exists a curve $\gamma$ : $[0,1] \to G_\varepsilon$, $\gamma(0) = x + \varepsilon n^G(x)$, for which the derivative $\dot\gamma(0)$ for which the derivative $H$.

In the next lemma, it is convenient to assume that the domain of definition of the mapping $x \mapsto n^G(x)$ is the surface $G$.

LEMMA 10. *For all $x \in G$, $\varepsilon < \varepsilon_b$, the relation $L_{\varepsilon,x} = H_x$ holds.*

*Proof.* We fix an arbitrary point $x \in G$ and a number $\varepsilon < \varepsilon_b$. Without loss of generality, we assume that $x + \varepsilon n^G(x) = 0$. In this case, we simply write $L_{\varepsilon,x}$ for $L_\varepsilon$. We show that $L_\varepsilon$ is a subspace of codimension 1 of the space $H$.

Let $F_b$ be a continuous linear functional represented by an element $b$. We define the function

$$\Psi \ : \ G^{\varepsilon_b} \to \mathbb{R} \ , \quad \omega \mapsto F_b(\psi(\omega)) \ . \tag{1.13}$$

Then, the set $G_\varepsilon$ can be written as $G_\varepsilon = \{\omega \in G^{\varepsilon_b} \ : \ \Psi(\omega) = \varepsilon\}$. Note that by continuity of the mapping $\psi$, the set $G^{\varepsilon_b}$ is open. The derivative $\Psi'(\omega)$ (in the topology of the space $H$) is nonzero at each point $\omega \in G^{\varepsilon_b}$. Indeed,

$$\Psi'(\omega)h = F_b(\psi'(\omega)h) = (b, \psi'(\omega)h) = ((\psi'(\omega))^* b, h) \ . \tag{1.14}$$

But the mapping $(\psi'(\omega))^*$ is injective because the mapping $\psi'(\omega)$ is bijective. Hence, we have $(\psi'(\omega))^* b \neq 0$.

In the space $H$, we consider the surface $G_\varepsilon \bigcap H$. Since $\Psi'(\omega)$ is nonzero at each point $\omega \in G^{\varepsilon_b}$, the tangent space at each point of this surface (in particular, at the point $\omega = 0$) is a subspace of codimension 1 in the space $H$. In particular, $L_\varepsilon$ is a subspace of codimension 1 in the space $H$.

Let us show that $L_\varepsilon = H_x$. Let $\tau \in L_\varepsilon$ $\gamma \ : \ [0,1] \to G_\varepsilon \bigcap H$ be a curve such that $\gamma(0) = 0$, $\dot\gamma(0) = \tau$ (the differentiability at the zero point in the topology of the space $H$). We set $\gamma_1(t) = P_{_G}(\gamma(t))$ and rewrite $\gamma_1(t)$ as $\gamma_1(t) = \varphi(P_b\psi(\gamma(t)), 0)$. This implies that $\gamma_1(t) \in H$ for all $t \in [0,1]$,and, by Lemma 6, the mapping $\gamma_1(\cdot)$ is differentiable at the point $t = 0$ in the topology of the space $H$. We have $\gamma(t) = \gamma_1(t) + \varepsilon n^{G}(\gamma_1(t))$. Multiplying this relation by $n^{G}(x)$, differentiating with respect to $t$ at the point $t = 0$, and using Lemmas 6 and 5, we obtain

$$(\tau, n^{G}(x)) = (\dot\gamma(0), n^{G}(x)) = (\dot\gamma_1(0), n^{G}(x)) + \frac{1}{2}\frac{d}{dt}\|n^{G}(\gamma_1(t))\|^2 \Big|_{t=0} = 0 \ ,$$

i.e., we have proved that $\tau \in H_x$. But both spaces $L_\varepsilon$ and $H_x$ are of codimension 1, and hence, they coincide. □

LEMMA 11. *There exist an $\varepsilon_b' \in (0, \varepsilon_b)$ and a constant $\lambda > 0$, such that, for all $\omega \in G^{\varepsilon_b'}$, the following estimate holds:*

$$|(b, \psi'(\omega)b)| > \lambda \ . \tag{1.15}$$

*Proof.* In notations of Lemma 8, we have

$$\psi'(x + tn^{G}(x)) = (A_0 + \Delta A_t)^{-1} = A_0^{-1} - A_0^{-1}\Delta A_t(A_0 + \Delta A_t)^{-1} \ .$$

It follows from the proof of Lemma 8, that $\|\Delta A_t\| < K_n t$ and $\|(A_0 + \Delta A_t)^{-1}\| < K_\psi$. We have the estimate:

$$|(b, \psi'(x + tn^{G}(x))b)| \geqslant |(b, \psi'(x)b)| - |(b, A_0^{-1}\Delta A_t(A_0 + \Delta A_t)^{-1}b)| \ . \tag{1.16}$$

Now we estimate the first term in (1.16). Let $\psi'(x)b = h$. Then $\varphi'(x, 0)h = b$, or

$$-k(x)(n^{G}(x), h)b + (b, h)n^{G}(x) + h = b, \qquad \text{where} \quad k(x) = \sqrt{1 + \|f'(x)\|^2} \ . \tag{1.17}$$

It follows from the form of this equation that we must search its solution in the form $h = \alpha n^G(x) + \beta b$ where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Substituting this expression in equation (1.17), we obtain the following system of equations for $\alpha$ and $\beta$

$$\begin{cases} \alpha = -\dfrac{1}{k(x) + (1+p(x))(1-p(x)k(x))} \\ \beta = -\alpha(1+p(x)) \ , \end{cases}$$

where $p(x) = (b, n^G(x))$. As already proved, $\varphi'(x,0)$ is a bijection. Hence the obtained solution is unique. Note that

$$(b, \psi'(x)b) = (b, \alpha n^G(x) + \beta b) = \alpha p(x) - \alpha(1+p(x)) = -\alpha \ .$$

Taking into account the inequalities $0 < k(x) < \sqrt{1+K_f^2}$ and $0 < p(x) \leqslant 1$, we obtain the estimate:

$$|k(x) + (1+p(x))(1-p(x)k(x))| \leqslant k(x) + (1+p(x))(1+p(x)k(x)) \leqslant \\ \leqslant \sqrt{1+K_f^2} + 2(1+\sqrt{1+K_f^2}) \stackrel{\text{den}}{=} \frac{1}{2\lambda} \ ,$$

where $\lambda$ is a constant. From this, we have

$$|(b, \psi'(x)b)| > 2\lambda \ . \tag{1.18}$$

Now we estimate the second term in (1.16).

$$|(b, A_0^{-1}\Delta A_t(A_0+\Delta A_t)^{-1}b)| \leqslant \|A_0^{-1}\|\|\Delta A_t\|\|(A_0+\Delta A_t)^{-1}\| \leqslant K_\psi^2 K_n t \ .$$

We choose $t_0 > 0$ such that $K_\psi^2 K_n t_0 < \lambda$. Then, taking into account (1.18), we see that (1.15) holds for all $0 < t < t_0$. We set $\varepsilon_b'$ to be equal to this value of $t_0$. □

Without loss of generality, we assume that $\varepsilon_b' = \varepsilon_b$.

LEMMA 12. *The set $G_\varepsilon$ is the graph of a continuous function $g_\varepsilon : \mathcal{U}(\varepsilon) \to \mathbb{R}$ defined on a set $\mathcal{U}(\varepsilon)$ open in $T_b$ and having the first-order derivative $(g_\varepsilon)' : \mathcal{U}(\varepsilon) \to H' \cong H$ and the second-order derivative $(g_\varepsilon)'' : \mathcal{U}(\varepsilon) \to \mathcal{L}(H, \mathcal{L}(H,H))$ along the subspace $H$, Moreover, there exists a constant $K_g > 0$ such that $\|g_\varepsilon''(x)\| < K_g$ for all $x \in \mathcal{U}(\varepsilon)$.*

*Proof.* Suppose that, as before, the function $\Psi$ is defined by formula (1.13). We apply Theorem 8 to the function $\Psi - \varepsilon$. Earlier it was proved that the mapping $\psi'$ is continuous, and hence the mapping $\Psi'$ is also continuous. As we have already proved, $\Psi'(\omega)$ is nonzero at each point $\omega \in G^{\varepsilon_b}$. Further, by Lemma 10, $n^{G^{\varepsilon_b}}(\omega) = n^G(x)$, where $x = P_b P_G(\omega)$. Therefore, by Remark 5, $(n^{G^{\varepsilon_b}}(\omega), b) > 0$

for all $\omega \in G^{\varepsilon_b}$. By Theorem 8, there exists a continuous function $g_\varepsilon$ defined on some open set $\mathcal{U}(\varepsilon) \subset T_b$, having first- and second-order derivatives along the subspace $H$, and such that $G_\varepsilon$ is the graph of this function (the fact that the function $\Psi$ also has a second-order derivative along the subspace $H$, follows from formula (1.14)).

Now we verify condition 4 of Theorem 8, which implies that $\|g''_\varepsilon(x)\|$ is bounded on $\mathcal{U}(\varepsilon)$ by a constant $K_g$. By Remark 5 and Lemma 10 for all $\omega \in G^{\varepsilon_b}$, we have

$$(b, n^{G^{\varepsilon_b}}(\omega)) > \frac{1}{\sqrt{1+K_f^2}} > 0 \ .$$

Further, by (1.14) and Lemma 8, we have $\|\Psi''(\omega)\| < K_\psi$ for all $\omega \in G^{\varepsilon_b}$.

Finally, we show that there exists a constant $\lambda > 0$ such that $\|\Psi'(\omega)\| > \lambda$. Since $|(b, n^{G^{\varepsilon_b}}(\omega))| \leqslant 1$ and $\|\Psi'(\omega)\||(b, n^{G^{\varepsilon_b}}(\omega))| = |\Psi'(\omega)b|$, then it suffices to prove that $|\Psi'(\omega)b| > \lambda$. We have $\Psi'(\omega)b = F_b(\psi'(\omega)b) = (b, \psi'(\omega)b)$, and the statement of this lemma follows from Lemma 11. □

LEMMA 13. *On the set* $\mathcal{U}\bigcap\mathcal{U}(\varepsilon)$ *the function* $g_\varepsilon$ *has the form*

$$g_\varepsilon(x) = f(x) + \frac{\varepsilon}{(n^G(x), b)} + \varepsilon^2\alpha_\varepsilon(x) \ ,$$

*moreover, there exists a constant* $C > 0$ *such that* $|\alpha_\varepsilon(x)| < C$ *for all* $\varepsilon \in (0, \varepsilon_b)$.

*Proof.* We choose an arbitrary point $x_0 \in \mathcal{U}\bigcap\mathcal{U}(\varepsilon)$. We set $x_1 = P_b(g_f(x_0) + \varepsilon n^G(x_0))$ and consider the restriction of the function $g_\varepsilon$ to the straight line passing through the points $x_0$ and $x_1$. We define the function $\tilde{g}_\varepsilon(t) = g_\varepsilon(x_1 + t\frac{x_0-x_1}{\|x_0-x_1\|})$ (for those $t \in \mathbb{R}$ for which the function $\tilde{g}_\varepsilon$ is well defined). The function $\tilde{g}_\varepsilon$ is defined and twice differentiable at some neighborhoods of the points $t = 0$ and $t = t_0 \stackrel{\text{def}}{=} \|x_0 - x_1\|$, and its second-order derivative is bounded by a constant $K_g$ by Lemma 12. Further, there exists a function which coincides with $\tilde{g}_\varepsilon$ at some neighborhoods of the points $t = 0 \quad t = t_0$, defined on the whole interval $[0, t_0]$, twice differentiable on its domain of definition (i.e. at a neighborhood of the interval $[0, t_0]$), and such that its second-order derivative is bounded by $K_g$. We will denote this function also by $\tilde{g}_\varepsilon$. Applying the Taylor formula to the function $\tilde{g}_\varepsilon$, we obtain

$$\tilde{g}_\varepsilon(t_0) = \tilde{g}_\varepsilon(0) + \tilde{g}'_\varepsilon(0)t_0 + \frac{1}{2}\tilde{g}''_\varepsilon(\theta t_0)t_0^2$$

where $\theta \in (0, 1)$.

Through the point $(x_1, g_\varepsilon(x_1))$, we draw the tangent $k(t) = \tilde{g}_\varepsilon(0) + \tilde{g}'_\varepsilon(0)t$ to the graph of the function $\tilde{g}_\varepsilon$. By Lemma 10, we have

$$k(t_0) - f(x_0) = \frac{\varepsilon}{(b, n^G(x_0))} \ . \tag{1.19}$$

Taking into account the relation $\tilde{g}_\varepsilon(t_0) = g_\varepsilon(x_0)$, we obtain

$$g_\varepsilon(x_0) - f(x_0) = (k(t_0) - f(x_0)) + (g_\varepsilon(x_0) - k(t_0)) \ , \tag{1.20}$$

where

$$|g_\varepsilon(x_0) - k(t_0)| = |\frac{1}{2}\tilde{g}''_\varepsilon(\theta t_0)\|x_0 - x_1\|^2| \leqslant \frac{1}{2}K_g\varepsilon^2 \ .$$

The statement of the lemma now follows from this, (2.8) and (2.9). $\square$

LEMMA 14. *Let $B \subset G$ be a Borel set, which is contained in $G$ together with its closure. together with its closure. Then there exists a constant $K > 0$ such that, for any $x \in T_b$, and for any $\varepsilon < \varepsilon_b$, the set $(x + R_b) \bigcap \bar{B}_\varepsilon$ ($\bar{B}_\varepsilon$ is the closure of $B_\varepsilon$) is contained in an interval of the line $x + R_b$ of length less than $K\varepsilon$.*

*Proof.* It is clear that it suffices to prove the statement of the lemma for a closed set $B$. We set $G_{\varepsilon,B} \stackrel{\text{def}}{=} \varphi(P_bB, \varepsilon)$. The line $x + R_b$ has a nonempty intersection with $\bar{B}_\varepsilon$ if and only if $x \in P_bG_{\varepsilon,B} \bigcup P_bB$. If $x \in P_bG_{\varepsilon,B} \bigcap P_bB$, then the statement of the lemma follows from Lemma 13.

Let $x \in P_bB \diagdown P_bG_{\varepsilon,B}$ (if the set $P_bB \diagdown P_bG_{\varepsilon,B}$ is empty, then we do not consider this case). The subset $(x + R_b) \bigcap \bar{B}_\varepsilon$ of the line $x + R_b$ is closed and bounded; hence it contains its least upper bound $\bar{\omega}$. It is clear that $\bar{t} \stackrel{\text{def}}{=} \|P_{_G}(\bar{\omega}) - \bar{\omega}\| < \varepsilon$. The intersection of the set $\bar{B}_\varepsilon$ with the line $x + R_b$ is the interval $[\varphi(x, 0), \bar{\omega}]$. Its length is equal to $g_{\bar{t}}(x) - f(x)$. The statement of the lemma again follows from Lemma 13 applied to the function $g_{\bar{t}}$.

Now let $x \in P_bG_{\varepsilon,B} \diagdown P_bB$, and let $\bar{\omega} \stackrel{\text{def}}{=} \inf\{(x + R_b) \bigcap \bar{B}_\varepsilon\}$ (if the set $P_bG_{\varepsilon,B} \diagdown P_bB$ is empty, then the lemma is proved). The intersection of the set $\bar{B}_\varepsilon$ with the line $x + R_b$ is the interval $[\bar{\omega}, x + g_\varepsilon(x)b]$. Let us estimate its length. Let $y \stackrel{\text{def}}{=} P_b(P_{_G}(\bar{\omega}) + \varepsilon n^G(P_bP_{_G}(\bar{\omega})))$. We set $e = \frac{x-y}{\|x-y\|}$ and define the function $g_e(t) = g_\varepsilon(y + te)$ $(t \in \mathbb{R})$. The function $g_e$ might not be defined for all values in the interval $t \in [0, t_x]$, where $t_x \stackrel{\text{def}}{=} \|x - y\|$. In this case, as in the proof of Lemma 13, by the function $g_e(t)$ we understand a function which coincides with $g_\varepsilon(y + te)$ at some neighborhoods of the points $t = 0$ and $t = t_x$ and which is extended on $[0, t_x]$ without increasing of the supremum of the second-order derivative.

Let $k_e$ be the tangent to the graph of the function $g_e(t)$ at the point $t = 0$. By Lemma 10, the tangent $k_e$ is orthogonal to the vector $n^G(P_bP_{_G}(\bar{\omega}))$. Then, taking into account the inequality $(b, n^G(x)) > 1/\sqrt{1 + K_f^2}$, we obtain the estimate

$$\|x + k_e(t_x)b - \bar{\omega}\| = \frac{\|\bar{\omega} - y + g_\varepsilon(y)b\|}{b, n^G(x))} < K_1\,\varepsilon, \tag{1.21}$$

where $K_1$ is an appropriate constant. By the Taylor formula with remainder in Lagrange form, there exists $t_0 \in (0, t_x)$ such that

$$|g_e(t_x) - k_e(t_x)| = |\frac{1}{2} g_e''(t_0) t_x^2| \leqslant \frac{1}{2} K_g \varepsilon^2 .$$

Hence, by (1.21), we have

$$\|x + g_\varepsilon(x)b - \bar{\omega}\| \leqslant \|x + k_e(t_x)b - \bar{\omega}\| + |g_e(t_x) - k_e(t_x)| < K\varepsilon,$$

where $K$ is an appropriate constant; we also took into account the fact that $g_e(t_x) = g_\varepsilon(x)$. □

*Proof of Theorem 9.* On the set $\mathcal{U} \times \mathbb{R}$, we define the function

$$f(x, \varepsilon) = f(x) + \frac{\varepsilon}{(n^G(x), b)} .$$

Let $B \subset G$ be an open set with property $(*)$. We set $f_\varepsilon(x) = f(x, \varepsilon)$, and

$$B_{f,f_\varepsilon} \stackrel{\text{def}}{=} \{x + tb \ , \ x \in P_b B \ , \ f(x) \leqslant t < f_\varepsilon(x)\} . \tag{1.22}$$

Applying Theorem 9 of [36] (obviously, the theorem is applicable in our case of a locally convex space, and one can easily check that the conditions of this theorem formulated in [36] are satisfied), we get

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \nu(B_{f,f_\varepsilon}) = \int_B (n^G(x), b) \frac{\partial f_\varepsilon}{\partial \varepsilon}(x, 0) \nu_G(dx) = \nu_G(B) . \tag{1.23}$$

Let $B_\varepsilon$ be a set of the form (1.11). If we prove that

$$\nu(B_\varepsilon \triangle B_{f,f_\varepsilon}) = o(\varepsilon) \tag{1.24}$$

for $\varepsilon < \varepsilon_b$, then we obtain the relation

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \nu(B_\varepsilon) = \nu_G(B) .$$

Changing the normal vector field to the opposite and taking into account the fact that Lemma 9 implies $\nu(B) = 0$, we prove (1.12).

Now we prove (1.24). Let $G_{\varepsilon,B} \stackrel{\text{def}}{=} \varphi(P_b B, \varepsilon)$. We set

$$M_1 \stackrel{\text{def}}{=} (B_\varepsilon \triangle B_{f,f_\varepsilon}) \bigcap (P_b B \cap P_b G_{\varepsilon,B} + R_b) ;$$
$$M_2 \stackrel{\text{def}}{=} (B_\varepsilon \triangle B_{f,f_\varepsilon}) \bigcap (P_b B \setminus P_b G_{\varepsilon,B} + R_b) ;$$
$$M_3 \stackrel{\text{def}}{=} (B_\varepsilon \triangle B_{f,f_\varepsilon}) \bigcap (P_b G_{\varepsilon,B} \setminus P_b B + R_b) .$$

Obviously, $M_1 \bigcup M_2 \bigcup M_3 = B_\varepsilon \triangle B_{f,f_\varepsilon}$. The set $G_{\varepsilon,B}$ is open (in $G_\varepsilon$) as the pre-image of the open (in $T_b + \varepsilon b$) set $P_b B + \varepsilon b$ under the continuous mapping $\psi$. By continuity of the function $g_\varepsilon$, the set $P_b G_{\varepsilon,B}$ is open (in $T_b$), hence each of the sets $M_1$, $M_2$, $M_3$, is a Borel set. Let us estimate the measure of each of them. To estimate $\nu(M_1)$, we note that $M_1$ has the form

$$M_1 = \{x+tb \ : x \in P_b B \bigcap P_b G_{\varepsilon,B} \ , \min\{f_\varepsilon(x), g_\varepsilon(x)\} \leqslant t < \max\{f_\varepsilon(x), g_\varepsilon(x)\}\}$$

(the function $g_\varepsilon$ is defined in Lemma 13). By Lemma 13 we have

$$\max\{f_\varepsilon(x), g_\varepsilon(x)\} - \min\{f_\varepsilon(x), g_\varepsilon(x)\} < C\varepsilon^2$$

where $C$ is a constant. Using Lemma 9, we obtain

$$\nu(M_1) \leqslant 4|d_b\nu|(P_b B \cap P_b G_{\varepsilon,B} + R_b)C\varepsilon^2 \leqslant 4|d_b\nu|(P_b B + R_b)C\varepsilon^2 \ . \tag{1.25}$$

Now, let us estimate $\nu(M_2)$. We set

$$M_{21} \stackrel{\text{def}}{=} (B_{f,f_\varepsilon} \setminus B_\varepsilon) \bigcap (P_b B \setminus P_b G_{\varepsilon,B} + R_b) \ ;$$
$$M_{22} \stackrel{\text{def}}{=} (B_\varepsilon \setminus B_{f,f_\varepsilon}) \bigcap (P_b B \setminus P_b G_{\varepsilon,B} + R_b) \ .$$

Clearly, the sets $M_{21}$ and $M_{22}$ are Borel sets, and $M_{21} \bigcup M_{22} = M_2$. First, we estimate $\nu(M_{21})$. Taking into account that $M_{21} \subset B_{f,f_\varepsilon}$ and $\sup_x |f_\varepsilon(x) - f(x)| < C_1\varepsilon$ (where $C_1$ is an appropriate constant) and using Lemma 9, we get

$$\nu(M_{21}) \leqslant 4_1 \, |d_b\nu|(P_b B \setminus P_b G_{\varepsilon,B} + R_b)\varepsilon \ . \tag{1.26}$$

Now we estimate $\nu(M_{22})$. For $x \in P_b B \setminus P_b G_{\varepsilon,B}$, we consider the subset $(x + R_b) \bigcap \bar{B}_\varepsilon$ of the line $x + R_b$. It is closed and bounded, and hence, it contains its least upper bound. On $P_b B \setminus P_b G_{\varepsilon,B}$ we define the function

$$h_\varepsilon(x) = f(x) + \sup\{(x + R_b) \bigcap \bar{B}_\varepsilon\} \ .$$

By Lemma 14, we have $\sup_x |h_\varepsilon(x) - f(x)| < K\varepsilon$ for all $\varepsilon < \varepsilon_b$.Then, taking into account $M_{22} \subset B_\varepsilon$, and using Lemma 9, we obtain

$$\nu(M_{22}) \leqslant 4K \, |d_b\nu|(P_b B \setminus P_b G_{\varepsilon,B} + R_b)\varepsilon \ . \tag{1.27}$$

Let us show that

$$|d_b\nu|(P_b B \setminus P_b G_{\varepsilon,B} + R_b) \to 0 \quad (\varepsilon \to 0) \ . \tag{1.28}$$

For this, it suffices to prove that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b B \setminus P_b G_{\varepsilon,B}}(x) = 0\ , \quad x \in T_{x_0} \tag{1.29}$$

($\mathbb{I}$ is the indicator function of the set in the subscript). Indeed, it follows from (1.29) that $\lim_{\varepsilon\to 0} \mathbb{I}_{P_b B \setminus P_b G_{\varepsilon,B}+R_b}(x) = 0$ ($x \in E$). From this and Lebesgue's theorem, we get (1.28).

To prove (1.29), it suffices to show that, for any point $x \in P_b B$, there exists $\bar{\varepsilon} > 0$ such that $x \in P_b G_{\varepsilon,B}$ for all $\varepsilon < \bar{\varepsilon}$ . We choose an arbitrary point $x \in P_b B$. The set $(x + R_b) \bigcap B^\varepsilon$ is an open subset of the line $x + R_b$. On the line $x + R_b$, we choose a neighborhood $U_{\omega_0}$ of the point $\omega_0 = x + f(x)b$ so that it is contained in $(x + R_b) \bigcap B^\varepsilon$. We set $\omega(t) = x + (f(x) + t)b$, $t \in \mathbb{R}$. Let $\bar{t} > 0$ be such that $\omega(\bar{t}) \in U_{\omega_0}$. On the interval $[0, \bar{t}]$, we define the function $\xi(t) = \|\omega(t) - P_G(\omega(t))\|$. The function $\xi(t)$ is continuous on the interval $[0, \bar{t}]$. Indeed, the function $\xi$ has the form $\xi(t) = \|P_b\psi(\omega(t)) - \psi(\omega(t))\|$, i.e. it is the composition of two functions $\psi(\omega(t))$ and $k(\omega) = \|P_b\omega - \omega\|$. The function $\psi(\omega(t))$ is continuous as a composition of continuous functions. Let us show that $k(\omega)$ is continuous. We write the argument $\omega$ in the form $\omega = (z, t)$ where $z \in T_b$ and $t \in \mathbb{R}$ ($E \simeq T_b \times \mathbb{R}$). Then the function $k$ can be rewritten as $k : (z, t) \mapsto t$, which is obviously continuous. The function $\xi(t)$ is continuous as a composition of continuous functions. We choose the value $\bar{\varepsilon}$ to be equal to $\xi(\bar{t})$. Clearly, $x \in P_b G_{\bar{\varepsilon},B}$. But the continuous function $\xi(t)$ takes all intermediate values from 0 to $\bar{\varepsilon}$ on the interval $[0, \bar{t}]$. Hence, $x \in P_{x_0} G_{\varepsilon,B}$ for all $\varepsilon < \bar{\varepsilon}$. The validity of (1.29) is proved.

Let us now estimate $\nu(M_3)$. For $x \in P_b G_{\varepsilon,B} \setminus P_b B$, we define the function

$$h_{1\varepsilon}(x) = g_\varepsilon(x) - \inf\{(x + R_b) \bigcap \bar{B}_\varepsilon\}\ .$$

It follows from Lemma 14 that $\sup_x |h_{1\varepsilon}(x) - g_\varepsilon(x)| < K\varepsilon$ for all $\varepsilon < \varepsilon_b$. Using Lemma 9, we get

$$\nu(M_3) \leqslant 4K|d_b\nu|(P_b G_{\varepsilon,B} \setminus P_b B + R_b)\,\varepsilon\ . \tag{1.30}$$

Let us show that

$$|d_b\nu|(P_b G_{\varepsilon,B} \setminus P_b B + R_b) \to 0 \quad (\varepsilon \to 0)\ . \tag{1.31}$$

We set $B_1 = \mathcal{U} \setminus \bar{B}$, (here $\bar{B}$ is the closure of $B$). The set $B_1$ is open (in $G$). By what was proved,

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b B_1 \setminus P_b G_{\varepsilon,B_1}}(x) = 0\ , \quad x \in T_b\ .$$

Hence

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_b B_1 \setminus P_b \bar{G}_{\varepsilon,B_1}}(x) = 0\ , \quad x \in T_b\ .$$

Then, taking into account that $(P_bG_{\varepsilon,B} \cap \mathcal{U}) \setminus P_b\bar{B} \subset P_bB_1 \setminus P_b\bar{G}_{\varepsilon,B_1}$, we get

$$\lim_{\varepsilon\to 0} \mathbb{I}_{(P_bG_{\varepsilon,B} \bigcap \mathcal{U})\setminus P_b\bar{B}} = 0 \ . \tag{1.32}$$

Further, let us show that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_bG_{\varepsilon,B}\setminus P_b\bar{B}} = 0 \ . \tag{1.33}$$

For this, by (1.32), it suffice to prove that

$$\lim_{\varepsilon\to 0} \mathbb{I}_{P_bG_{\varepsilon,B}\setminus \mathcal{U}} = 0 \ . \tag{1.34}$$

everywhere on $T_b$. We choose an arbitrary point $x_0 \in T_b \setminus \mathcal{U}$. We assume that the convergence (1.34) does not hold at the point $x_0$. Then there exists a sequence $\varepsilon_n \to 0$, such that $x_0 \in P_bG_{\varepsilon_n,B} \setminus \mathcal{U}$ for all $n$. Without loss of generality one can consider that $x_0 \in H$. Define $G^H_{\varepsilon_n,B} = G_{\varepsilon_n,B} \bigcap H$, $\mathcal{U}^H = \mathcal{U} \bigcap H$. We have

$$x_0 \in P_bG^H_{\varepsilon_n,B} \setminus \mathcal{U}^H \ . \tag{1.35}$$

On the other hand, $\mathcal{U}^H$, being open with respect to the induced topology of the space $H_b$, is open with respect to its Hilbert topology. Similarly, the set $\bar{B}^H = \bar{B} \bigcap H$ is closed in the Hilbert space $H_b$. Note that the set $P_bG^H_{\varepsilon_n,B}$ is contained in the $\varepsilon_n$-neighborhood of the set $\bar{B}^H$, and, since $\bar{B}^H \subset \mathcal{U}^H$, all such $\varepsilon_n$-neighborhoods, starting from some $n$, are contained in $\mathcal{U}^H$, which contradicts (1.35). Thus, (1.34) holds, and hence, (1.33) holds too.

For the indicator functions of arbitrary sets $A_1$ and $A_2$, the following equality holds:

$$\mathbb{I}_{A_1\setminus A_2} = \mathbb{I}_{A_1} - \mathbb{I}_{A_2} + \mathbb{I}_{A_2\setminus A_1} \ . \tag{1.36}$$

We have

$$P_bG_{\varepsilon,B} \setminus P_b\bar{B} = (P_bG_{\varepsilon,B} \setminus P_bB) \setminus P_b\partial B \ .$$

Applying (1.36) to the sets $P_bG_{\varepsilon,B} \setminus P_bB$ and $P_b\partial B$, we get

$$\mathbb{I}_{P_bG_{\varepsilon,B}\setminus P_b\bar{B}} = \mathbb{I}_{P_bG_{\varepsilon,B}\setminus P_bB} - \mathbb{I}_{P_b\partial B} + \mathbb{I}_{P_b\partial B\setminus(P_bG_{\varepsilon,B}\setminus P_bB)} \ .$$

Hence,

$$\mathbb{I}_{P_bG_{\varepsilon,B}\setminus P_b\bar{B}+R_b} = \mathbb{I}_{P_bG_{\varepsilon,B}\setminus P_bB+R_b} - \mathbb{I}_{P_b\partial B+R_b} + \mathbb{I}_{P_b\partial B\setminus(P_bG_{\varepsilon,B}\setminus P_bB)+R_b} \ . \tag{1.37}$$

Since $\nu_{_T}(P_b\partial B) = 0$ (property $(*)$), then $\nu(\partial B + R_b) = 0$, and hence,

$$|d_b\nu|(\partial B + R_b) = 0 \ . \tag{1.38}$$

From (1.37), it follows that

$$|d_b\nu|(P_bG_{\varepsilon,B}\setminus P_b\bar{B}+R_b)=|d_b\nu|(P_bG_{\varepsilon,B}\setminus P_bB+R_b)-\\ -|d_b\nu|(\partial B+R_b)+|d_b\nu|((\partial B+R_b)\setminus((P_bG_{\varepsilon,B}\setminus P_bB)+R_b))\ ,$$

where, by (1.38), the last two terms are zero. From the $\sigma$-additivity of the measure $|d_b\nu|$, Lebesgue's theorem, and the equality (1.33), we get (1.31).

Now from the estimates (1.25), (1.26), (1.27), (1.30), and from the validity of (1.28) and (1.31) it follows that the equality (1.24) holds. □

Assume that the function $f$ can be extended to some domain $\mathcal{U}_1\subset T_b$ such that $\bar{\mathcal{U}}\subset\mathcal{U}_1$ and the graph of the extended function is the surface in the sense of Definition 5. We denote by $\Sigma$ the set, which consists of all open subsets of the surface $G$ possessing property $(*)$. Let $\nu^\varepsilon$ be a measure on $G$, defined as follows: $\nu^\varepsilon(B)=\frac{\nu(B^\varepsilon)}{2\varepsilon}$.

THEOREM 10. *Let the family of functions* $f_B(\varepsilon)=\frac{\nu^\varepsilon(B)}{\nu_{_G}(B)}$, *where* $B\in\Sigma$, *be uniformly bounded on the interval* $(0,\varepsilon_b]$. *Then, formula* (1.12) *holds for every Borel subset of the surface* $G$.

*Proof.* From the assumptions imposed on the function $f$, it follows that the equality (1.12) holds for open subsets of the surface $G$, that possess property $(*)$.

Now let $B\subset G$ be an arbitrary open set. We use the fact that $\nu$ is a Radon measure. According to [35], in this case $\nu_{_G}$ is also a Radon measure on $G$. We find a $\sigma$-compact $K_1\subset B$, and a $\sigma$-compact $\tilde{K}_2\subset B^{\varepsilon_b}$, such that $\nu_{_G}(K_1)=\nu_{_G}(B)$ $\nu(\tilde{K}_2)=\nu(B^{\varepsilon_b})$. Let $K_2=P_{_G}(\tilde{K}_2)$ . By continuity of the mapping $P_{_G}$, $K_2$ is a $\sigma$-compact. We set $K=K_1\bigcup K_2$. It is clear that $K$ is a $\sigma$-compact $K\subset B$, $\nu_{_G}(K)=\nu_{_G}(B)$ and $\nu(K^{\varepsilon_b})=\nu(B^{\varepsilon_b})$. For each point $x\in K$, we choose its neighborhood $\mathcal{V}_x\subset B$, which possesses property $(*)$ and satisfies the conditions $\nu^{\varepsilon_b}(\partial\mathcal{V}_x)=0$ and $\nu_{_G}(\partial\mathcal{V}_x)=0$ (we take into account the fact that the measures $\nu^{\varepsilon_b}$ and $\nu_{_G}$ are $\sigma$-additive on $G$). From the open covering of the $\sigma$-compact $K$ by the sets $\mathcal{V}_x$, we choose a countable subcovering $\mathcal{V}_i$ $(i=1,2,\ldots)$. Define

$$\mathcal{F}_1=\mathcal{V}_1\ ,\quad \mathcal{F}_i=\mathcal{V}_i\setminus\bigcup_{k=1}^{i-1}\mathcal{V}_k\ ,\quad i=2,3,\ldots\ .$$

Clearly, a finite union of open subsets of the surface $G$ possessing property $(*)$, is an open subset of the surface $G$ possessing property $(*)$. It is easy to see that for a Borel set of the form $A_1\setminus A_2$, where $A_1$ and $A_2$ are open subsets of the surface $G$ which possess the property $(*)$, equality (1.12) holds. Hence, for all sets $\mathcal{F}_i$ $(i=1,2,\ldots)$, equality (1.12) holds. We set $\mathcal{F}=\bigcup_i\mathcal{V}_i$. The sets $\mathcal{F}_i$ do not intersect and $\bigcup\limits_i \mathcal{F}_i = \mathcal{F}$. We show that (1.12) also holds for the set $\mathcal{F}$. We have

$$\lim_{\varepsilon\to 0} \frac{\nu(\mathcal{F}^\varepsilon)}{2\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \sum_{i=1}^{\infty} \nu(\mathcal{F}_i^\varepsilon) = \lim_{\varepsilon\to 0} \sum_{i=1}^{\infty} \nu^\varepsilon(\mathcal{F}_i) \ . \tag{1.39}$$

It follows from the assumptions of the theorem that the series $\sum\limits_{i=1}^{\infty} \nu^\varepsilon(\mathcal{F}_i)$ can be majorized by the converging series $\sum\limits_{i=1}^{\infty} \nu_{G}(\mathcal{F}_i)$ multiplied by some positive constant and hence converges uniformly. Then, we can pass to the limit under the sum sign in (1.39). Taking into account $\sigma$-additivity of the measure $\nu_G$, we get

$$\nu_G(\mathcal{F}) = \lim_{\varepsilon\to 0} \frac{\nu(\mathcal{F}^\varepsilon)}{2\varepsilon} \ .$$

From the construction of the set $\mathcal{F}$, it follows that $\nu_G(\mathcal{F}) = \nu_G(B)$ and $\nu(\mathcal{F}^{\varepsilon_b}) = \nu(B^{\varepsilon_b})$. The last equality implies that $\nu(\mathcal{F}^\varepsilon) = \nu(B^\varepsilon)$ for all $\varepsilon \leqslant \varepsilon_b$. We have

$$\nu_G(B) = \nu_G(\mathcal{F}) = \lim_{\varepsilon\to 0} \frac{\nu(\mathcal{F}^\varepsilon)}{2\varepsilon} = \lim_{\varepsilon\to 0} \frac{\nu(B^\varepsilon)}{2\varepsilon} \ .$$

We have proved (1.12) for open subsets of the surface $G$. Now let $B \subset G$ be a closed set. Since the equality (1.12) holds for the surface $G$ itself, which is an open subset of the surface $G_1$, and for its open subset $G \diagdown B$, this relation also holds for the set $B$.

Now suppose that $B \subset G$ is an arbitrary Borel set. Let $K_n$ be a sequence of compact sets such that $K_n \subset B$, $\nu_G(K_n) \to \nu_G(B)$, $n \to \infty$, and $A_n$ be is a sequence of open sets such that $B \subset A_n$ and $\nu_G(A_n) \to \nu_G(B)$, $n \to \infty$. Both sequences exists because $\nu_G$ is a Radon measure. For each fixed $n$, we have

$$\nu_G(K_n) = \lim_{\varepsilon\to 0} \frac{\nu(K_n^\varepsilon)}{2\varepsilon} \leqslant \varliminf_{\varepsilon\to 0} \frac{\nu(B^\varepsilon)}{2\varepsilon} \leqslant \varlimsup_{\varepsilon\to 0} \frac{\nu(B^\varepsilon)}{2\varepsilon} \leqslant \lim_{\varepsilon\to 0} \frac{\nu(A_n^\varepsilon)}{2\varepsilon} = \nu_G(A_n) \ .$$

Passing to the limit in this system of inequalities as $n \to \infty$, we obtain (1.12). □

## 1.3 Application to Gaussian measures

Let $E$ be a Hilbert space, $\gamma$ a centered Gaussian Radon measure on $E$ with the correlation operator $B$. The scalar product in $E$ we denote by $(\cdot,\cdot)_E$, and the corresponding norm, by $\|\cdot\|_E$. The scalar product $H$, we denote by $(\cdot,\cdot)$, as before. Let $H(\gamma)$ be the Cameron-Martin space of the measure $\gamma$. We will also assume $b \in B(E)$. The norm of the space $H$ can be expressed via the norm of the space $E$ in the following way:

$$\|h\| = \sup\left\{(l,h)_E \ : l \in E, \sqrt{\int_X (l,x)_E^2 \gamma(dx)} \leqslant 1\right\}$$
$$= \left\{(\sqrt{B}l, (\sqrt{B})^{-1}h)_E \ : l \in E, \|\sqrt{B}l\|_E \leqslant 1\right\} = \|(\sqrt{B})^{-1}h\|_E \ ,$$

and for all $x \in E$ and $y \in H$ the scalar product $(\cdot,\cdot)_E$ can be expressed via the scalar product $(\cdot,\cdot)$ as follows:

$$(x,y)_E = (Bx,y) = (\sqrt{B}x, \sqrt{B}y) \ .$$

THEOREM 11. *Suppose that the following conditions are satisfied:*

1) *The domain $\mathcal{U}$ is a bounded set and the function $f$ is bounded on $\mathcal{U}$;*

2) *the derivative $f''(x)$ is a Hilbert-Schmidt operator at each point $x \in T_b$, and there exists a constant $M > 0$ such that $\|f''(x)\|_1 < M$ for all $x \in T_b$, where $\|\cdot\|_1$ is the Hilbert-Schmidt norm;*

3) *there exist centered Gaussian measures $\gamma_1$ and $\gamma_2$ on $T_b$ and $R_b$ respectively, such that $\gamma \sim \gamma_1 \otimes \gamma_2$, and there exist constants $N_1$ $N_2$, such that $N_1 \leqslant \frac{d\gamma}{d(\gamma_1\otimes\gamma_2)} \leqslant N_2$ on $G + R_b$ (in particular, $\gamma = \gamma_1 \otimes \gamma_2$);*

4) *the divergence $\delta\tau$ of the vector field $\tau(x) = P_b n^G(x)$ with respect to the measure $\gamma_1$ is bounded from above on $\mathcal{U}$ by some constant $M_\tau$;*

5) *$(B^{-1}b, b)_E > 0$.*

*Then the family $f_Q(\varepsilon) = \frac{\gamma^\varepsilon(Q)}{\gamma_G(Q)}$, $Q \in \mathfrak{B}(G)$, is uniformly bounded, and, consequently, the formula* (1.12) *holds for any Borel subset of the surface $G$.*

Note that, by (1.3) and condition 2.2, $\tau$ is the mapping of the class $W^{2,1}(\gamma_1, H_b)$ (defined in [7]). In this case, according to [7], it has the divergence $\delta\tau$.

For a proof of the theorem we need the next lemma.

LEMMA 15. *For all $Q \in \mathfrak{B}_G$, the following inequality holds:*

$$\gamma_G(Q) \geqslant \lambda |d_b\gamma|(Q + R_b) \ , \tag{1.40}$$

*where $\lambda > 0$ is a constant.*

*Proof.* We have

$$d_b\gamma = -(B^{-1}b, \cdot)_E \cdot \gamma \ . \tag{1.41}$$

Set $L_b = \{\omega \in E \ : \ (B^{-1}b, \omega)_E = 0\} = \{(x,t) \in T_b \times \mathbb{R} \ : \ t = -\frac{(B^{-1}b,x)_E}{(B^{-1}b,b)_E}\}$ .

Since $(B^{-1}b, b)_E > 0$, then the set of the points located under the hypersubspace $L_b$ (with respect to the axis $R_b \cong \mathbb{R}$) has a positive $d_b\gamma$-measure. We denote this set by $E^+$. Suppose first that $\operatorname{gr} f \subset E^+$. Let $M_f = \sup_{\mathcal{U}} f$. Let $Q \subset G$ be an arbitrary Borel subset. We introduce the following notations:

$$\tilde{Q} \stackrel{\text{def}}{=} \bigcup_{s\leqslant 0}(Q + sb) \ , \quad Q_1 \stackrel{\text{def}}{=} \bigcup_{s\leqslant -2M_f} (P_bQ + sb) \ ,$$

$$Q_2 \stackrel{\text{def}}{=} \left\{x + tb \ : \ x \in P_bB, \ -2M_f \leqslant t \leqslant -\frac{(B^{-1}b, x)_E}{(B^{-1}b, b)_E}\right\} \ .$$

We have

$$d_b\gamma(\tilde{Q}) \geqslant d_b\gamma(Q_1) \ . \tag{1.42}$$

Further, by the definition of the surface measure [35],

$$\gamma_G(Q) = \int\limits_Q \frac{\gamma^d(d\omega)}{(n^G(P_b\omega), b)_E} \ , \quad \text{where } \gamma^d(Q) = d_b\gamma(\tilde{Q}) \ .$$

From this, it follows that

$$\gamma_G(Q) \geqslant d_b\gamma(\tilde{Q}) \ . \tag{1.43}$$

Further, we will show that where exists a constant $\lambda > 0$ such that

$$d_b\gamma(Q_1) \geqslant \lambda\, d_b\gamma(Q_1 \cup Q_2) \ . \tag{1.44}$$

This, by the inequalities (1.42) and (1.43), yields that $\gamma_G(Q) \geqslant \frac{\lambda}{2} d_b\gamma(Q_1 \cup Q_2)$. Note that $|d_b\gamma|(Q + R_b) = 2d_b\gamma(Q_1 \cup Q_2)$. This, by the last inequality, implies (1.40). To prove (1.44), we first show that there exists a constant $K > 0$ such that

$$d_b\gamma(Q_1) \geqslant K\gamma(Q_1) \ . \tag{1.45}$$

We introduce the following notations: $W_1 = \bigcup_{s\leqslant -2C}(\bar{\mathcal{U}} + sb)$, $\quad a = -\frac{B^{-1}b}{\|B^{-1}b\|}$. By (1.41), it suffices to prove that for all $\omega \in W_1$, the inequality $|(B^{-1}b, \omega)_E| \geqslant K$ holds, or, which is the same,

$$|(a, \omega)_E| \geqslant \tilde{K} \quad \text{for all } \omega \in W_1 \tag{1.46}$$

and for a constant $\tilde{K}$. Note that the quantity $|(a, \omega)_E|$ is the distance from the point $\omega$ to the hypersubspace $L_b$. Since the inequality $-\frac{(B^{-1}b,x)_E}{(B^{-1}b,b)_E} \geqslant f(x) \geqslant -M_f$, holds on $\bar{\mathcal{U}}$, then $L_b \bigcap W_1 = \varnothing$. From this and from the fact that the sets $L_b$ and

$W_1$ are closed, it follows that $\rho(L_b, W_1) > 0$, where the metric $\rho$ is generated by the norm of the space $E$. Choosing $\tilde{K} = \rho(L_b, W_1)$, we obtain that (1.46) holds, and hence (1.45) holds too.

Further,

$$(\gamma_1 \otimes \gamma_2)(Q_1) = \gamma_1(P_b Q)\gamma_2(-\infty, -2M_f] = \gamma_1(P_b Q)C_1 \tag{1.47}$$

where $C_1 = \gamma_2(-\infty, -2M_f]$.

Let $\ell$ denote the Lebesgue measure on the real line. Define $Q_3 = \cup_{-2M_f \leqslant s \leqslant 0}(P_b Q + sb)$, $Q_4 = Q_2 \triangle Q_3$. By Fubini's theorem,

$$(\gamma_1 \otimes \ell)(Q_4) = \int_{P_b Q} \left| \frac{(B^{-1}b, x)_E}{(B^{-1}b, b)_E} \right| \gamma_1(dx) \ .$$

Since the set $\mathcal{U}$ is bounded, then the function $\left|\frac{(B^{-1}b,x)}{(B^{-1}b,b)}\right|$ is bounded on $\mathcal{U}$ by some constant $C_2$. Hence

$$(\gamma_1 \otimes \ell)(Q_4) \leqslant C_2 \gamma_1(P_b Q) \ . \tag{1.48}$$

Since the measure $\gamma_2$ is absolutely continuous with respect to the Lebesgue measure and has a positive density $p(t)$ bounded on each bounded set, then, by Fubini's theorem, the measure $\gamma_1 \otimes \gamma_2$ is absolutely continuous with respect to the measure $\gamma_1 \otimes \ell$ and has the same density. Using this fact and the inequality (1.48), we get $(\gamma_1 \otimes \gamma_2)(Q_4) \leqslant C_3 \gamma_1(P_b Q)$, where $C_3$ is a positive constant. Further,

$$\begin{aligned}(\gamma_1 \otimes \gamma_2)(Q_2) &\leqslant (\gamma_1 \otimes \gamma_2)(Q_3) + (\gamma_1 \otimes \gamma_2)(Q_4) \\ &\leqslant \gamma_1(P_b Q)\gamma_2[-2M_f, 0] + C_2\gamma_1(P_b Q) = C_3\gamma_1(P_b Q) \ ,\end{aligned}$$

where $C_4 = \gamma_2[-2M_f, 0] + C_3$. From this and (1.47), it follows that

$$\frac{(\gamma_1, \otimes \gamma_2)(Q_1)}{(\gamma_1 \otimes \gamma_2)(Q_2)} \geqslant K_1 \ , \tag{1.49}$$

where $K_1$ is a constant. We show that there exists a constant $K_2 > 0$ such that

$$d_b\gamma(Q_2) \leqslant K_2\gamma(Q_2) \tag{1.50}$$

We have $d_b\gamma(Q_2) = -\int_{Q_2}(B^{-1}b, \omega)_E \gamma(d\omega)$. Taking into account that the set

$$W_2 = \left\{ x + tb \ : \ x \in \mathcal{U}; -2M_f \leqslant t \leqslant -\frac{(B^{-1}b, x)_E}{(B^{-1}b, b)_E} \right\}$$

is bounded and $Q_2 \subset W_2$, we get that the function $-(B^{-1}b, \omega)_E$ is bounded on $Q_2$ by some positive constant $K_2$, which does not depend on choice of the set $Q \subset G$. This implies (1.50).

Condition 3 of Theorem 11, and the inequalities (1.44), (1.49), (1.50) now give us

$$d_b\gamma(Q_1) \geqslant K\gamma(Q_1) \geqslant K\,N_1(\gamma_1\otimes\gamma_2)(Q_1) \geqslant K\,N_1\,K_1(\gamma_1\otimes\gamma_2)(Q_2) \geqslant K\,N_1\,K_1\,N_2\,\gamma(Q_2) \geqslant \frac{KN_1K_1N_2}{K_2}\,d_b\gamma(Q_2)\ ,$$

which implies (1.44) for the case when $\mathrm{gr}\, f \subset E^+$. It is clear that (1.44) holds also for the case $\mathrm{gr}\, f \subset E^-$. Let us consider the general case of location of $\mathrm{gr}\, f$ with respect to the hypersubspace $L_b$. As before, we let $Q \subset G$ be a Borel set. Define $Q' = Q \cap E^+$, $Q'' = Q \diagdown Q'$. It follows from (1.44) that there exists $\lambda > 0$ such that

$$\gamma_G(Q') \geqslant \lambda|d_b\gamma|(Q' + R_b)\ , \quad \gamma_G(Q'') \geqslant \lambda|d_b\gamma|(Q'' + R_b)\ ,$$

which implies (1.44) for the general case. □

*Proof of Theorem 11.* Let $Q \subset G$ be a Borel set. Let $\varepsilon_0 > 0$, $\varepsilon \in [0, \varepsilon_0]$ , and let $f(\cdot, \varepsilon)\ :\ G \to R_b$ be a Borel function such that $f(\cdot, 0) = f$. Applying the results of the paper [36] to our situation, gives us the following estimate:

$$|\varepsilon^{-1}\nu(Q_{f,f_\varepsilon}) - \nu_{_G}(Q)| \leqslant C\varepsilon|d_bd_b\nu|(Q + R_b) \tag{1.51}$$

where $Q_{f,f_\varepsilon}$ is the set of the form (2.4), and in our case the constant $C$ does not depend on choice of $Q \in \mathcal{B}(G)$. For a proof of the uniform boundedness of the family $f_Q(\varepsilon)$, it suffices to prove the uniform boundedness of the values

$$\varepsilon^{-1}\gamma(Q_{f,f_\varepsilon})\diagdown\gamma_G(Q) \quad \text{and} \quad \varepsilon^{-1}\gamma(Q^\varepsilon \bigtriangleup Q_{f,f_\varepsilon})\diagdown\gamma_G(Q)$$

Using the inequalities (1.25), (1.26), (1.27), (1.30), (1.51), and Lemma 15, we obtain that

$$\left|\frac{\varepsilon^{-1}\gamma(Q_{f,f_\varepsilon})}{\gamma_{_G}(Q)} - 1\right| \leqslant C\varepsilon\,\frac{|d_bd_b\gamma|(Q + R_b)}{|d_b\gamma|(Q + R_b)}\ , \tag{1.52}$$

$$\frac{\varepsilon^{-1}\gamma(M_2)}{\gamma_{_G}(Q)} \leqslant 4C\varepsilon\ , \tag{1.53}$$

$$\frac{\varepsilon^{-1}\gamma(M_2)}{\gamma_{_G}(Q)} \leqslant 4C\,\frac{|d_b\gamma|(P_bQ\diagdown P_bG_{\varepsilon,Q} + R_b)}{|d_b\gamma|(Q + R_b)}\ , \tag{1.54}$$

$$\frac{\varepsilon^{-1}\gamma(M_3)}{\gamma_{_G}(Q)} \leqslant 4C\,\frac{|d_b\gamma|(P_bG_{\varepsilon,Q}\diagdown P_bQ + R_b)}{|d_b\gamma|(Q + R_b)} \tag{1.55}$$

where the constant $C$ does not depend on the set $Q \in \Sigma$. Let us show the uniform boundedness of the right-hand sides of these inequalities. For the inequalities (1.53) and (1.54) it is obvious. We show this uniform boundedness for the inequality (1.52). For an arbitrary set $Q \in \mathcal{B}_E$, we have

$$d_b d_b \gamma(Q) = -d_b \int_Q (B^{-1}b,\omega)_E \gamma(d\omega) = \int_Q \left((B^{-1}b,\omega)_E^2 - (B^{-1}b,b)_E\right) \gamma(d\omega)\ . \tag{1.56}$$

Let $E = E^{++} \cup E^{--}$ be a Hahn-Jordan decomposition with respect to the measure $d_b d_b \gamma$. It is clear that $E^{--} = \{\omega \in E\ :\ (B^{-1}b,\omega)_E^2 - (B^{-1}b,b)_E < 0$. Presenting $\omega \in E$ in the form $\omega = x + tb$, $x \in T_b$, $t \in \mathbb{R}$, we get

$$E^{--} = \Big\{(x,t) \in E\ :\ \frac{(B^{-1}b,x)_E}{(B^{-1}b,b)_E} - \frac{1}{\sqrt{(B^{-1}b,b)_E}} < t < -\frac{(B^{-1}b,x)_E}{(B^{-1}b,b)_E} + \frac{1}{\sqrt{(B^{-1}b,b)_E}}\Big\}$$

Let $Q \in \mathcal{B}_G$, $Q^+ = (Q+R_b)^+$, $Q_- = (Q+R_b)^-$, where $Q + R_b = (Q+R_b)^+ \cup (Q+R_b)^-$ is a Hahn-Jordan decomposition with respect to the measure $d_b d_b \gamma$. We have

$$|d_b d_b \gamma|(Q + Rb) = d_b d_b(Q_+) - |d_b d_b \gamma|(Q_-) = -2 d_b d_b \gamma(Q_-)\ . \tag{1.57}$$

We will show that there exists a constant $C_1 > 0$ such that

$$-d_b d_b \gamma(Q_-) \leqslant C_1 |d_b \gamma|(Q_-)\ . \tag{1.58}$$

Define $Q_{-+} = Q_- \cap E^+$, $Q_{--} = Q_- \cap E^-$, $\tilde{G} = (G + R_b) \cap E^+ \cap E^{--}$, where $E = E^+ \cup E^-$ is a Hahn-Jordan decomposition with respect to $d_b\gamma$. From (1.56) it follows that

$$d_b d_b \gamma(Q_{-+}) = -\int_{Q_{-+}} (B^{-1}b,\omega)_E\, d_b\gamma(d\omega) - (B^{-1}b,b)\gamma(Q_{-+})\ .$$

From this, by boundedness of the function $|(B^{-1}b,\omega)_E|$ on $\tilde{G}$ (by some constant $C_2 > 0$), we get

$$-d_b d_b \gamma(Q_{-+}) = |d_b d_b \gamma(Q_{-+})| \leqslant C_2 d_b \gamma(Q_{-+}) + (B^{-1}b,b)_E \gamma(Q_{-+})\ . \tag{1.59}$$

Define

$$Q'_{-+} = \Big\{(x,t) \in Q_{-+}\ :\ t < -\frac{(B^{-1}b,x)_E}{(B^{-1}b,b)_E} - \frac{1}{2\sqrt{(B^{-1}b,b)_E}}\Big\}\ .$$

We show that there exists a constant $C_3 > 0$ such that

$$\gamma(Q_{-+}) \leqslant C_3\, \gamma(Q'_{-+})\ . \tag{1.60}$$

Since the measure $\gamma$ is absolutely continuous with respect to the measure $\gamma_1 \times \ell$, and since, by boundedness of the set $\tilde{G}$, there exist positive constants $K_1$ and $K_2$ such that for any Borel subset $A \subset \tilde{G}$

$$K_1\,(\gamma_1 \otimes \ell)(A) \leqslant \gamma(A) \leqslant K_2\,(\gamma_1 \otimes \ell)(A)\ .$$

Then, to prove (1.60), it suffices to prove that

$$(\gamma_1 \otimes \ell)(Q_{-+}) \leqslant C_3(\gamma_1 \otimes \ell)(Q'_{-+})$$

for some positive constant $C_3$. But the last inequality is obvious because by Fubini's theorem, the measure $\gamma_1 \otimes \ell$ is invariant with respect to shifts along the line $R_b$, and consequently

$$(\gamma_1 \otimes \ell)(Q_{-+}) = 2(\gamma_1 \otimes \ell)(Q'_{-+})\ .$$

Thus, (1.60) is proved. Setting $a = -\frac{B^{-1}b}{\|B^{-1}b\|}$, for all $\omega \in Q'_{-+}$, we obtain that

$$-(B^{-1}b, \omega)_E = \|B^{-1}b\|_E\,(a, \omega)_E \geqslant \|B^{-1}b\|_E\,\rho(L_b, Q'_{-+}) = \frac{\sqrt{(B^{-1}b, b)_E}}{2}\ ,$$

where the metric $\rho$ is generated by the norm of the space $E$. From this it follows that

$$(Q'_{-+}) \leqslant C_4\, d_b\gamma(Q'_{-+})$$

where $C_4 = \frac{\sqrt{(B^{-1}b,b)_E}}{2}$. Taking into account the last inequality and the inequality (1.60), we get

$$\gamma(Q_{-+}) \leqslant C_3\gamma(Q'_{-+}) \leqslant C_3C_4\, d_b\gamma(Q'_{-+}) \leqslant C_3C_4\, db(Q_{-+})\ .$$

From this and (1.59), it follows that there exists a constant $C_1 > 0$, which does not depend on the set $Q \subset G$ and such that

$$-d_bd_b\gamma(Q_{-+}) \leqslant C_1 d_b\gamma(Q_{-+})\ . \tag{1.61}$$

In exactly the same way, we can show that

$$-d_bd_b\gamma(Q_{-+}) \leqslant C_1(-d_b\gamma)(Q_{-+})\ .$$

This together with (1.61) implies (1.58). Further, using (1.57), we get

$$|d_bd_b\gamma|(Q + R_b) = -2d_bd_b\gamma(Q_-) \leqslant C_1|d_b\gamma|(Q_-) \leqslant C_1|d_b\gamma|(Q + R_b)\ ,$$

which implies that $\frac{|d_bd_b\gamma|(Q+R_b)}{|d_b\gamma(Q+R_b)}$ is bounded. Finally, we show that the right-hand side of inequality (1.55) is uniformly bounded. As before, we let $Q_1 = \bigcup_{s\leqslant -2M_f}(P_bQ + sb)$. Using inequality (1.45), we obtain

$$\begin{aligned}|d_b\gamma|(Q + R_b) &= 2d_b\gamma(Q_+) \geqslant 2d_b\gamma(Q_1) \geqslant 2K\gamma(Q_1) \geqslant C_5\gamma_1 \otimes \gamma_2(Q_1)\\ &= C_5\gamma_1(P_bQ)\gamma_2(-\infty, -2M_f) = C_6\gamma_1(P_bQ)\gamma_2(-\infty, \infty) = C_7\gamma_1(P_bQ)\ ,\end{aligned} \tag{1.62}$$

where $C_5$, $C_6$ $C_7$ are constants, which does not depend on $Q \in \Sigma$. Further, taking into account the fact that the function $(B^{-1}b, x)_E$ is bounded on $\mathcal{U}$, for an arbitrary $Q \in \mathfrak{B}_G$ we get

$$|d_b\gamma|(Q + R_b) = 2d_b\gamma(Q_+) = \int_{Q_+} (B^{-1}b, \omega)_E(d\omega)$$
$$\leqslant -N_2 \int_{Q_+} ((B^{-1}b, x)_E + t(B^{-1}b, b)_E)\gamma_1 \otimes \gamma_2(d(x,t)) \leqslant C_8\gamma_1(P_bQ)\gamma_2(\mathbb{R})$$
$$+ (B^{-1}b, b)_E \int_{P_bQ} \gamma_1(dx) \int_{-\infty}^{-\frac{(B^{-1}b,x)_E}{(B^{-1}b,b)_E}} |t|\gamma_2(dt) \leqslant C_9\gamma_1(P_bQ)\ ,$$

where $C_8$, $C_9$ are constants, which does not depend on the set $Q \in \mathfrak{B}_G$. From this and from inequalities (1.55) and (1.62), it follows that

$$\frac{\varepsilon^{-1}\gamma(M3)}{\gamma_G(Q)} \leqslant N\,\frac{\gamma_1(P_bG_{\varepsilon,Q} \setminus P_bQ)}{\gamma_1(P_bQ)}\ , \tag{1.63}$$

where $N$ is a constant which does not depend on $Q \in \Sigma$.

Let $T_\varepsilon\ : \mathcal{U} \to T_b$, $x \to x + \varepsilon\tau(x)$. We denote by $\tilde{\gamma}_1$ the measure $\gamma_1 \circ T_\varepsilon^{-1}$, i.e. the image of $\gamma_1$ under to the mapping $T_b$. The measure $\tilde{\gamma}_1$ is proved to be absolutely continuous with respect to the measure $\gamma_1$, and its density $\frac{d\tilde{\gamma}_1}{d\gamma_1}$ is bounded from zero on $\mathcal{U}$. This follows from the next theorem [7]: Let $X$ be a LCS, $\gamma_0$ a Gaussian measure on $X$, $T = I + F$ where $F\ : X \to H(\gamma_0)$ is a $\gamma_0$-measurable mapping of the class $W^{2,1}(\gamma_0, H(\gamma_0))$ (defined in [7]), such that

$$\|F(x+h) - F(x)\|_{H(\gamma_0)} \leqslant \lambda\|h\|_{H(\gamma_0)} \quad \text{for all} \quad h \in H(\gamma_0) \quad \gamma_0\text{– a.e.}, \tag{1.64}$$

where $\lambda < 1$. Suppose that the stochastic derivative $D_{H(\gamma_0)}F(x)$ is a Hilbert-Schmidt operator for a.e. $x$, and, a.e., $\|D_{H(\gamma_0)}F(x)\|_2 < M_F < \infty$. Then, the density of the measure $\gamma_0 \circ T^{-1}$ with respect to the measure $\gamma_0$ has the form

$$\frac{d(\gamma_0 \circ T^{-1})}{d\gamma_0}(x) = \frac{1}{\Lambda_F(T^{-1}(x))}\ ,$$

where $\Lambda_F(x) = \Big|\det_2\big(I + D_{H(\gamma_0)}F(x)\big)\exp\big[\delta F(x) - \frac{1}{2}\|F(x)\|^2_{H(\gamma_0)}$, $\delta F$ is a divergence of the vector field $F$ with respect to the measure $\gamma_0$, $det_2$ is the regularized Fredholm-Carleman determinant (which are defined in [7]).

In our case $X = T_b$, $\gamma_0 = \gamma_1$, $F = \varepsilon\tau$, $T = T_\varepsilon$. Note that the stochastic derivative $D_{H(\gamma_0)}\tau$ is equal to the derivative $\tau'$ of the mapping $\tau$ along the subspace $H(\gamma_1) = H_b$. Obviously, it suffices to check that inequality (1.64) holds for the mapping $\tau$ and for an arbitrary positive constant $\lambda$. Note that $\tau'(x) = P_b(n^G)'(x)$, and, consequently, by Lemma (5), $\|\tau'(x)\| < K_n$ everywhere on $T_b$. Hence for all $h \in H_b$, the inequality $\|\tau(x+h) - \tau(x)\| < K_n h$ holds. Further, from (1.3) and condition 2, it follows that for all $x \in T_b$, $(n^G)(x)$ is a Hilbert-Schmidt operator, and there exists a constant $M_n > 0$ such that $\|(n^G)(x)\|_{(2)} < M_n$. Since $\tau(x) = n^G(x) - (b, n^G(x))b$, then for all $x \in T_b$, $\tau'(x)$ is also the Hilbert-Schmidt operator and $\|\tau'(x)\|_{(2)} \leqslant 2\|(n^G)'(x)\|_{(2)} < 2M_n$. Thus, for the mapping $\varepsilon\tau$, all the conditions of the above theorem are satisfied. Hence $\frac{d(\gamma_1 \circ T^{-1})}{d\gamma_1}(x) = \frac{1}{\Lambda(T_\varepsilon^{-1}(x))}$. Let us show that $\Lambda_F$ bounded on $\mathcal{U}$. By the Carleman inequality [7],

$$|\det{}_2(I + \varepsilon\tau'(x))| \leqslant \exp\left[\frac{\varepsilon}{2}\|\tau'(x)\|_{(2)}^2\right] < e^{2M_n^2\varepsilon_b}$$

This together with condition 4 of the theorem implies the boundedness of $\Lambda_F$ on $\mathcal{U}$, say by some constant $M_\Lambda$. Further, taking into account that $T_\varepsilon(P_b Q) = P_b G_{\varepsilon,Q}$, we get

$$\gamma_1(P_b Q) = \gamma_1 \circ T^{-1}(P_b G_{\varepsilon,Q}) = \int_{P_b G\varepsilon,Q} \frac{\gamma_1(dx)}{\Lambda_F(T_\varepsilon^{-1}(x))} \geqslant \frac{1}{M_\Lambda}\gamma_1(P_b G_{\varepsilon,Q})$$

From this inequality and inequality (1.63), it follows that

$$\frac{\varepsilon^{-1}\gamma(M_3)}{\gamma_G(Q)} \leqslant M_\Lambda \frac{\gamma_1(P_b G_{\varepsilon,Q} \diagdown P_b Q)}{\gamma_1(P_b G_{\varepsilon,Q})} \leqslant M_\Lambda \ .$$

The last estimate completes the proof of the theorem. □

## 1.4 Construction of the surface measure

Definition 6. *We call the set $G$ a **surface in a wide sense** if each point $x \in G$ has a neighborhood $U_x \subset G$, which is a surface in the sense of Definition 5.*

Let $G$ be a surface in a wide sense. We want to define the function $\nu^G$ on all Borel subsets of the surface $G$. In what follows, the symbols $\varepsilon_x$, $P_x$, $x$ and $T_x$ denote $\varepsilon_b$, $P_b$, $\psi$, and $T_b$, respectively, considered relative to the surface $U_x$. On Borel subsets of each neighborhood $U_x$, we define a function $\nu^G$ by the formula

$$\nu^G(B) = \lim_{\varepsilon \to 0} \frac{\nu(B^\varepsilon)}{2\varepsilon} \ . \tag{1.65}$$

The measure $\nu^G$ is well defined. If $B \subset U_x \bigcap U_{x'}$, then formula (1.65) shows that the value of $\nu^G(B)$ is independent of choice of a neighborhood containing the set $B$.

Let $B \subset G$ be an arbitrary Borel set. Define $\tilde{B} \stackrel{\text{def}}{=} \bigcup_{x \in B} U_x^{\varepsilon_x}$. The set $\tilde{B}$ is open. Choose $\sigma$-compact $K \subset \tilde{B}$ such that $\nu(K) = \nu(\tilde{B})$. From the open cover of the

$\sigma$-compact $K$ with the sets $U_x^{\varepsilon_x}$ we choose a countable subcover. Hence one can represent the set $\tilde{B}$ as a union of a countable number of sets $U_{x_i}^{\varepsilon_{x_i}}$ $(i = 1, 2, \ldots)$ and a set $\tilde{C}$ of a zero-measure. Define $U_i = U_{x_i}$, $\varepsilon_i = \varepsilon_{x_i}$. Then the following decomposition holds:

$$\tilde{B} = \tilde{C} \bigcup \bigcup_i U_i^{\varepsilon_i} \ . \tag{1.66}$$

Consider the sets

$$V_1 = U_1 \ , \quad V_i = U_i \setminus \bigcup_{k=1}^{i-1} U_k \ , \quad i = 2, 3, \ldots \ .$$

They are mutually disjoint, and

$$\bigcup_i V_i = \bigcup_i U_i \ .$$

Define

$$B_i = V_i \bigcap B \ .$$

Since $B \subset \bigcup_i V_i$, and since $V_i$ are mutually disjoint, then we can represent the set $B$ in the form:

$$B = C \bigcup \bigcup_i B_i \ , \tag{1.67}$$

where $B_i$ are mutually disjoint, $B_i \subset V_i \subset U_i$, and $C = \tilde{C} \bigcap B$.

Let a Borel set $A_0 \subset G$ be such that there exists a Borel set $A \subset G$ containing $A_0$ as a subset, and such that for any $x \in A$ there exists $t_x > 0$ such that the set

$$\hat{A} = \{x + tn^G(x) \ , \ x \in A, \ t \in (-t_x, t_x)\} \tag{1.68}$$

is Borel and is of measure zero. Set $\nu^G(A_0) = 0$.

LEMMA 16. *The definition is correct in the following sense: if the set $A_0$ is contained in a neighborhood $U_{x_0}$. which is a surface in the sense of Definition 5, then $\nu^G(A_0) = \nu_G(A_0) = 0$.*

*Proof.* We can assume that the set $\hat{A}$ is contained in $U^{\varepsilon_{x_0}}$. If not, then we can consider its intersection with $U^{\varepsilon_{x_0}}$, which is also Borel and is of measure zero. Consider the sets

$$F(\varepsilon) = \{x \in A \ : \ t_x \geqslant \varepsilon\} \ , \quad \varepsilon < \varepsilon_{x_0} \ .$$

They are Borel. Indeed, the set $F(\varepsilon)$ is Borel if and only if the set $F'(\varepsilon) = P_{x_0} F(\varepsilon)$ is Borel. Let $\hat{A}' = \psi_{x_0}(\hat{A})$. The set $\hat{A}'$ is Borel and has the form

$$\hat{A}' = \{(x, t) \ : \ x \in A' \ , \ t \in (-t_x, t_x)\}$$

where $A' = P_{x_0}A$. Then the intersection of $\hat{A}'$ and the closed linear manifold $T_{x_0} + \varepsilon b$ is Borel and has the form

$$\{x \in A' \ : \ t_x \geqslant \varepsilon\} + \varepsilon b = F'(\varepsilon) + \varepsilon b \ .$$

Consequently, the set $F'(\varepsilon)$, and hence the set $F(\varepsilon)$ are also Borel.

Further, we have the inclusion $F(\varepsilon)^{\varepsilon} \subset \hat{A}$. Then for any $\varepsilon' < \varepsilon$, $\nu((F(\varepsilon)^{\varepsilon'}) = 0$, and hence $\nu^G(F(\varepsilon)) = 0$. One can present the set $A$ in the form

$$A = \bigcup_{n > \varepsilon_{x_0}^{-1}} F\left(\frac{1}{n}\right) ,$$

where, by what was proved, $\nu^G(F(n^{-1})) = 0$. Hence $\nu^G(A) = 0$, and since $A_0 \subset A$, then $\nu^G(A_0) = 0$. □

Consider decomposition (1.66) of the set $\tilde{B}$. Define

$$U_{ix} = U_i \bigcap U_x \ .$$

The sets $U_{ix}$ are open and

$$\bigcup_{i,\, x \in B} U_{ix} = \bigcup_i U_i \ , \tag{1.69}$$

$$\bigcup_i U_i^{\varepsilon_i} \subset \bigcup_{i,\, x \in B} U_{ix}^{\varepsilon_x} \ . \tag{1.70}$$

The set

$$\hat{C} \stackrel{\text{def}}{=} \tilde{B} \diagdown \bigcup_{i,\, x \in B} U_{ix}^{\varepsilon_x}$$

is Borel and has the form (1.68). By (1.70), it is a subset of the set $\tilde{C}$, and hence it is of measure zero. From (1.69), it follows that $\tilde{C} \cap G = \hat{C} \cap G$. But $\nu^G(\hat{C} \cap G) = 0$ , and, consequently, since $C \subset \tilde{C} \cap G$, $\nu^G(C) = 0$.

We define

$$\nu^G(B) = \sum_i \nu^G(B_i) \ .$$

LEMMA 17. *The function $\nu^G$ is well defined and is $\sigma$-additive.*

*Proof.* Indeed, suppose that we have another decomposition

$$B = C' \bigcup \bigcup_i B_i' \ ,$$

where $\nu^G(C') = 0$, the sets $B'_i$ are mutually disjoint, and each of them contained in some neighborhood $U_x$. For the set $B'_i$ we denote this neighborhood by $U'_i$. Define

$$B_{ij} = B_i \bigcap B'_j \ .$$

From $\sigma$-additivity of the measure $\nu^G$ on $U_i$ and $U'_i$, it follows that

$$\nu^G(B_i) = \sum_j \nu^G(B_{ij}) \ ,$$

$$\nu^G(B'_i) = \sum_j \nu^G(B_{ji}) \ .$$

Then, by the non-negativity of $\nu^G$, we have

$$\nu^G(B) = \sum_i \nu^G(B_i) = \sum_{i,j} \nu^G(B_{ij}) = \sum_j \nu^G(B'_j) \ .$$

□

DEFINITION 7. *We call the function $\nu^G$ a* ***surface measure*** *on the surface $G$.*

Since for all sets $B_i$ in the decomposition (1.67), $\nu^G(B_i) < \infty$, then the following statement holds.

STATEMENT 1. *The surface measure $\nu^G$ $\sigma$–finite.*

STATEMENT 2. *The surface measure $\nu^G$ possesses the Radon property, i.e.,*

$$\nu^G(B) = \sup\{\nu^G(K) \ : \ K \subset B, \, K \text{ — } \textit{compact}\} \ . \tag{1.71}$$

*Proof.* Note that from the non-negativity of the function $\nu^G$, it follows that

$$\nu^G(B) \geqslant \sup\{\nu^G(K) \ : \ K \subset B, \, K \text{ — compact}\} \ . \tag{1.72}$$

By what was proved, any Borel subset $B$ can be presented in the form:

$$B = C \bigcup \bigcup_i B_i \ ,$$

where $B_i$ are mutually disjoint Borel subsets, each of them is contained in some neighborhood $U_{x_i}$, and $\nu^G(C) = 0$.

According to [35], the function $\nu^G$ is a Radon measure on each neighborhood $U_x$. Let $\delta > 0$, and let $K_{i,\delta} \subset B_i$ be a compact such that $\nu(B_i \diagdown K_{i,\delta}) < \dfrac{\delta}{2^i}$. Define $\bar{K}_{n,\delta} = \bigcup\limits_{i=1}^{n} K_{i,\delta}$. Then, we have

$$\nu^G(B) = \sup_{n,\delta}\{\nu^G(\bar{K}_{n,\delta})\} \ .$$

This and (1.72) implies (1.71). □

# Chapter 2

# Application of the surface layer theorems to the proof of the Stokes formula

## 2.1 Operations in the class differential forms of the Sobolev type relative to a smooth measure on LCS

Let $H$ be a Hilbert space with the scalar product $(\,\cdot\,,\,\cdot\,)_H = (\,\cdot\,,\,\cdot\,)$, $\{e_n\}_{n=1}^{\infty}$be an orthonormal basis in $H$. Let $\Gamma(n)$ denote the set of all increasing sequences of natural numbers of length $n \in \mathbb{N}$, $\Gamma(0) = \{0\}$. If $\gamma_1 \in \Gamma(n)$, $\gamma_2 \in \Gamma(m)$, then we consider the sequences $\gamma_1 \cup \gamma_2$ and $\gamma_1 \setminus \gamma_2$ $(n \geqslant m)$ as elements of $\Gamma(m+n)$ and $\Gamma(n-m)$ respectively, putting them in the increasing order, if necessarily. For every $n \in \mathbb{N}$ and $\gamma = (i_1, \ldots, i_n) \in \Gamma(n)$, the symbol $e_\gamma$ denotes $e_{i_1} \wedge \ldots \wedge e_{i_n}$, $e_0 = 1$, where the vectors $e_{i_1}, \ldots, e_{i_n}$ are considered as linear continuous functionals on $H$ (the operation $\wedge$ is defined, for example, in [8]). By $L_n(H)$ we denote the space of all antisymmetric $n$-linear Hilbert-Schmidt functionals. Note that $L_n(H)$ is a Hilbert space with the scalar product $(g_1, g_2)_n = \sum\limits_{i_1<\cdots<i_n} g_1(e_{i_1}, \ldots, e_{i_n})\, g_2(e_{i_1}, \ldots, e_{i_n})$ (we denote the corresponding norm by $\|\cdot\|_n$), and $\{e_\gamma\}_{\gamma\in\Gamma(n)}$ is the orthonormal basis in $L_n(H)$. Note that if $f \in L_n(H)$, $g \in L_m(H)$, then $f \wedge g \in L_{m+n}(H)$. Indeed, denoting the sequence

of vectors $e_{i_1}, \dots, e_{i_n}$ also by the symbol $e_\gamma$ for simplicity, we get

$$\sum_{\gamma\in\Gamma(m+n)} (f\wedge g)^2(e_\gamma) = \sum_{\gamma\in\Gamma(m+n)} \left( \sum_{\gamma_1\in\Gamma(n)} \varepsilon(\sigma)\, f(e_{\gamma_1})\, g(e_{\gamma\setminus\gamma_1}) \right)^2 \leqslant$$
$$\leqslant C^n_{m+n} \sum_{\gamma\in\Gamma(m+n)} \sum_{\gamma_1\in\Gamma(n)} f^2(e_{\gamma_1})\, g^2(e_{\gamma\setminus\gamma_1}) = C^n_{m+n} \sum_{\substack{\gamma_1\in\Gamma(n),\\ \gamma_2\in\Gamma(m):\\ \gamma_1\cup\gamma_2=\gamma}} f^2(e_{\gamma_1})\, g^2(e_{\gamma_2}) \leqslant$$
$$\leqslant C^n_{m+n} \sum_{\gamma_1\in\Gamma(n)} f^2(e_{\gamma_1}) \sum_{\gamma_2\in\Gamma(m)} g^2(e_{\gamma_2}) < \infty\ .$$

Analogously to the finite dimensional case [32], for elements $f \in L_m(H)$ and $g \in L_n(H)$ $(m > n)$, one can define the element $g \lrcorner f \in L_{m-n}(H)$ by the equality $(g \lrcorner f, h)_{m-n} = (f, g\wedge h)_n$, which holds for all $h \in L_{m-n}(H)$. Show that the operation $\lrcorner$ is well defined for $f \in L_m(H)$ and $g \in L_n(H)$. Indeed, if the element $g \lrcorner f$ exists, then it uniquely defined by its coordinates in the basis $\{e_\gamma\}_{\gamma\in\Gamma(m-n)}$. We have to cheek that

$$\sum_{\gamma\in\Gamma(m-n)} (g \lrcorner f, e_\gamma)^2_{m-n} = \sum_{\gamma\in\Gamma(m-n)} (f, g\wedge e_\gamma)^2_m < \infty\ .$$

Note that

$$(g\wedge e_\gamma)(e_{\gamma'}) = \sum_{\substack{\gamma_1\in\Gamma(n),\\ \gamma_2\in\Gamma(m-n):\\ \gamma_1\cup\gamma_2=\gamma}} \varepsilon(\sigma)\, g(e_{\gamma_1})\, e_\gamma(e_{\gamma_2}) = \begin{cases} 0, & \gamma \not\subset \gamma' \\ \varepsilon(\sigma) g(e_{\gamma'\setminus\gamma}), & \gamma\subset\gamma' \end{cases}$$

where $\varepsilon(\sigma)$ is the permutation parity of $\sigma$. Here we also used the definition of exterior multiplication [8]. Taking into account the last inequality, we obtain

$$\sum_{\gamma\in\Gamma(m-n)} (f, g\wedge e_\gamma)^2_m = \sum_{\gamma\in\Gamma(m-n)} \left( \sum_{\gamma'\in\Gamma(m)} f(e_{\gamma'})\, (g\wedge e_\gamma)(e_{\gamma'}) \right)^2 =$$

$$= \sum_{\gamma\in\Gamma(m-n)} \left( \sum_{\substack{\gamma'\in\Gamma(m):\\ \gamma\subset\gamma'}} \varepsilon(\sigma)\, f(e_{\gamma'})\, g(e_{\gamma'\setminus\gamma}) \right)^2 \leqslant$$

$$\leqslant \sum_{\gamma\in\Gamma(m-n)} \left( \sum_{\substack{\gamma_1\in\Gamma(n):\\ \gamma\cap\gamma_1=\varnothing}} f^2(e_{\gamma\cup\gamma_1}) \sum_{\substack{\gamma_1\in\Gamma(n):\\ \gamma\cap\gamma_1=\varnothing}} g^2(e_{\gamma_1}) \right) \leqslant$$

$$\leqslant \sum_{\gamma_1\in\Gamma(n)} g^2(e_{\gamma_1}) \sum_{\substack{\gamma\in\Gamma(m-n),\\ \gamma_1\in\Gamma(n):\\ \gamma\cap\gamma_1=\varnothing}} f^2(e_{\gamma\cup\gamma_1}) \leqslant C_m^n \sum_{\gamma_1\in\Gamma(n)} g^2(e_{\gamma_1}) \sum_{\gamma_2\in\Gamma(m)} f^2(e_{\gamma_2}) < \infty\ .$$

Now let $X$ be a locally convex space, and a Hilbert subspace $H$ is a vector subspace of the space $X$.

Definition 8. *A mapping $f\ :\ X\to L_n(H)$ is called a differential form of a degree $n$ defined on $X$.*

Every differential form $f$ can be presented in the form: $f=\sum\limits_{\gamma\in\Gamma(n)} f_\gamma\, e_\gamma$, where $f_\gamma\ :\ X\to\mathbb{R}$ are functions. The operations of exterior and interior multiplications for differential forms are defined pointwise.

Let $f$ be a differential form of degree $n$.

Definition 9. *We say that $f$ possesses a differential if all its coefficients $f_\gamma$ are differentiable in each of the directions $e_p$ for $p\notin\gamma$, and for all $x\in X$, $\sum\limits_{\gamma\in\Gamma(n),\,p\notin\gamma} d_{e_p}f_\gamma(x)\, e_p\wedge e_\gamma\ \in\ L_n(H)$. A differential form $df$ of degree $n+1$ is called the differential of the differential form $f$ if*

$$df = \sum_{\gamma\in\Gamma(n),\,p\notin\gamma} d_{e_p}f_\gamma\, e_p\wedge e_\gamma\ .$$

Definition 10. *We say that $f$ possesses a codifferential if all its coefficients $f_\gamma$ are differential in each directions $e_p$ for $p\in\gamma$, and for all $x\in X$, $\sum\limits_{p\in\gamma\in\Gamma(n)} d_{e_p}f_\gamma(x)\, e_p\,\lrcorner\, e_\gamma\in L_n(H)$. A differential form $\delta f$ of degree $n-1$ is called the codifferential of the differential form $f$ if*

$$\delta f = \sum_{p\in\gamma\in\Gamma(n)} d_{e_p}f_\gamma\, e_p\,\lrcorner\, e_\gamma.$$

Further, let $\mathfrak{B}_X$ be the $\sigma$-algebra of Borel sets of the space $X$. By a measure, we understand a $\sigma$-additive function defined on $\mathfrak{B}_X$, with values in a Hilbert space.

Definition 11. *A $\sigma$-additive measure on $X$ and taking values in $L_n(H)$ is called a differential form of codegree $n$.*

Each differential form $\omega$ of codegree $n$ can be presented in the form $\omega = \sum_{\gamma\in\Gamma(n)} \omega_\gamma e_\gamma$, where $\omega_\gamma$ are real valued $\sigma$-additive measures.

Definition 12. *Let $g$ be a bounded differential form of degree $m$, $\omega$ be a differential form of codegree $n \geqslant m$, which is a measure of a bounded variation. A differential form $g \wedge \omega$ of codegree $n - m$ defined as*

$$(g \wedge \omega)(A) = \int_A g(x) \lrcorner\, \omega(dx)\ ,$$

*is called an exterior product of the differential forms $g$ and $\omega$.*

The form $g \wedge \omega$ is well defined. Indeed, it is a differential form of codegree $n - m$. Further, the differential form $\omega$ can be presented in the form $\omega = f \cdot |\omega|$ (see [10]), where $|\omega|$ denotes the variation of $\omega$, and $f$ is a differential form of degree $n$ such that $\|f(x)\|_n = 1$ for $|\omega|$-almost all $x$. We have

$$(g \wedge \omega)(A) = \int_A \big(g(x) \lrcorner\, f(x)\big) |\omega|(dx).$$

Further,

$$\sum_{\gamma\in\Gamma(n-m)} ((g \wedge \omega)(A), e_\gamma)^2_{n-m} = \sum_{\gamma\in\Gamma(n-m)} \Big(\int_A \big(g(x) \lrcorner\, f(x), e_\gamma\big)_{n-m} |\omega|(dx)\Big)^2 \leqslant$$

$$\leqslant |\omega|(A) \sum_{\gamma\in\Gamma(n-m)} \int_A \big(g(x) \lrcorner\, f(x), e_\gamma\big)^2_{n-m} |\omega|(dx) =$$

$$= |\omega|(A) \int_A \|g(x) \lrcorner\, f(x)\|^2_{n-m} |\omega|(dx) \leqslant C_n^m \big(|\omega|(A)\big)^2 \sup_x \|g(x)\|_m < \infty$$

Definition 13. *We say that a differential form $\omega$ of codegree $n$ possesses a differential if all its coefficients are differentiable in all directions $e_p$ for $p \in \gamma$, and $\sum_{p\in\gamma\in\Gamma(n)} d_{e_p}\omega_\gamma(A)\, e_p \lrcorner\, e_\gamma \in L_n(H)$ for all $A \in \mathfrak{B}_X$. A differential form $d\omega$ of codegree $n - 1$ defined as*

$$d\omega = (-1)^{n-1} \sum_{p\in\gamma\in\Gamma(n)} d_{e_p}\omega_\gamma\, e_p \lrcorner\, e_\gamma\ ,$$

*is called a differential of the differential form $\omega$.*

LEMMA 18. *Let $g$ $\omega$ are differential form of degree $m$ and codegree $n+1>m$ respectively, which possess differentials and are such that $g$ and $dg$ are bounded, $\omega$ and $d\omega$ are of a bounded variation. Then the differential form $g\wedge\omega$ possess a differential, and*

$$d(g\wedge\omega)=g\wedge d\omega+(-1)^n dg\wedge\omega\ . \tag{2.1}$$

Equality (2.1) is easily obtained. One should use definitions of differentials for $g$ and $\omega$, the definition of the operation $\wedge$, and compare coefficients at each $e_\gamma$.

Now we find the operator adjoint to the operator $d$ for some class of the Sobolev type differential forms relative to real or complex valued $\sigma$-additive measure on $X$. Let $\mu$ be such a measure. We assume that $\mu$ is differentiable in each direction $e_p$ [5], and for all $x$ $\|\beta^\mu(x)\|_H<\infty$, where $\beta^\mu(x)=\sum_p \beta^\mu_{e_p}(x)\,e_p$, and $\beta^\mu_{e_p}$ is the logarithmic derivative of the measure $\mu$ along the direction $e_p$ [5, 26]. Further let the numbers $p>1$ and $q>1$ be such that $1/p+1/q=1$. By $\Omega^n_p$, denote the vector space consisting of all differential $n$-forms $f$ such that $\int_X\|f(x)\|^p_n\,\mu(dx)<\infty$ Define a norm in $\Omega^n_p$ by setting

$$\|f\|_{n,p}=\left(\int\limits_X\|f(x)\|^p_n\,\mu(dx)\right)^{1/p}\ .$$

For the elements $f=\sum\limits_{\gamma\in\Gamma(n)} f_\gamma(x)\,e_\gamma\in\Omega^n_p$ and $\omega=\sum\limits_{\gamma\in\Gamma(n)}\omega_\gamma(x)\,e_\gamma\in\Omega^n_q$, we define a bilinear operation

$$\langle\omega,f\rangle_n=\int\limits_X\Big(\omega(x),f(x)\Big)_n\,\mu(dx)\ . \tag{2.2}$$

The integral on the right-hand side exists by the Hölder inequality and the definition of the normed space $\Omega^n_p$. By the definition of the scalar product $(\cdot,\cdot)_n$ and by the Lebesgue theorem, we rewrite (2.2) in the form

$$\langle\omega,f\rangle_n=\sum_{\gamma\in\Gamma(n)}\int\limits_X f_\gamma(x)\,\omega_\gamma(x)\,\mu(dx)\ . \tag{2.3}$$

Here we also took into account the absolute convergence of the series on the right-hand side of (2.3).

Let $A^n_p$ be a vector subspace of the vector space $\Omega^n_p$, consisting of those $f$, which possess a codifferential $\delta f$, and such that

$$\left(\int\limits_X\|\delta f(x))\|_{n-1}\,\mu(dx)\right)^{1/p}+\left(\int\limits_X\|\beta^\mu(x)\|^p_H\,\|f(x)\|^p_n\,\mu(dx)\right)^{1/p}<\infty \tag{2.4}$$

and satisfying the condition: for every $\gamma \in \Gamma(n)$, there exists $\delta > 0$ and non-negative functions $g_\gamma(x)$, $g_{1\gamma}(x)$, and $g_{2\gamma}(x)$, such that $g_\gamma(x)$ is $d_{e_p}\mu$-summable for every $p \notin \gamma$, $g^2_{1\gamma(x)}$ $g^2_{2\gamma}(x)$ is $\mu$-summable, and for all $p \notin \gamma$ for $|t| < \delta$ the inequalities $|f_\gamma(x+te_p)| < \min\{g_\gamma(x), g_{1\gamma}(x)\}$ and $|d_{e_p} f_\gamma(x+te_p)| < g_{2\gamma}(x)$ hold. Define a norm on $A^n_p$ by setting

$$\|f\|_{A^n_p} = \|f\|_{n,p} + \left(\int\limits_X \|\delta f(x)\|_{n-1}\,\mu(dx)\right)^{1/p} + \left(\int\limits_X \|\beta^\mu(x)\|^p_H \,\|f(x)\|^p_n\,\mu(dx)\right)^{1/p}.$$

Further let $B^n_q$ be a vector space of the space of the space $\Omega^n_q$ consisting of those $\omega$ possessing a differential $d\omega$ and satisfying the following condition: for every $\gamma \in \Gamma(n)$ there exist $dl > 0$ and non-negative functions $g_\gamma(x)$, $g_{1\gamma}(x)$, $g_{2\gamma}(x)$, such that $g^2_\gamma(x)$ $d_{e_q}\mu$-summable for all $q \in \gamma$, $g^2_{1\gamma}(x)$, $g^2_{2\gamma}(x)$ $\mu$-summable, and for all $q \in \gamma$, for $|t| < \delta$, the inequalities $|\omega_\gamma(x + t_q)| < \min\{g_\gamma(x), g_{1\gamma}(x)\}$ and $|d_{e_q}\omega_\gamma(x + te_q)| < g_{2\gamma}(x)$ hold. Define the norm on the space $B^n_q$:

$$\|\omega\|_{B^n_q} = \|\omega\|_{n,q} + \|d\omega\|_{n+1,q}\ .$$

It is clear that $d\ :\ B^n_q \to \Omega^{n+1}_q$ is a linear continuous operator Note that by the inequality $|\langle\omega, f\rangle_n| \leqslant \|\omega\|_{n,q}\,\|f\|_{n,p} \leqslant \|\omega\|_{n,q}\,\|f\|_{A^n_p}$. Every element $f \in A^n_p$ can be presented as a linear continuous functional on $\Omega^n_q$ defined as in (2.2). Analogously, by the inequality $|\langle\omega, f\rangle_n| \leqslant \|\omega\|_{B^n_q}\,\|f\|_{n,p}$ each element $f \in \Omega^n_p$ as a linear continuous functional on $B^n_q$. In this sense, one can speak about the adjoint operator $d^*\ :\ A^{n+1}_p \to \Omega^n_p$.

THEOREM 12. *For every pair of elements $f \in A^{n+1}_p$ and $\omega \in B^n_q$, where $1/p + 1/q = 1$, the element $-\beta^\mu \lrcorner\, f - \delta f$ belongs to $\Omega^n_p$. In addition to this,*

$$\langle d\omega, f\rangle_{n+1} = \langle\omega, -\beta^\mu \lrcorner\, f - \delta f\rangle_n\ ,$$

*i.e., on $A^{n+1}_p$, the adjoint operator $d^*$ has the form*

$$d^* = -(\beta^\mu \lrcorner + \delta)\ .$$

*Proof.* Let $\omega = \sum\limits_{\gamma\in\Gamma(n)} \omega_\gamma e_\gamma$, $f = \sum\limits_{\gamma_1\in\Gamma(n+1)} f_{\gamma_1} e_{\gamma_1}$. We have

$$\|\beta^\mu(x) \lrcorner\, f(x)\|_n =$$

$$= \left\| \sum_{p\in\gamma_1\in\Gamma(n+1)} \beta^\mu_{e_p}(x)\, f_{\gamma_1}(x)\, e_p \lrcorner\, e_{\gamma_1} \right\|_n = \left\| \sum_{\gamma\in\Gamma(n)} \left( \sum_{p\notin\gamma} \beta^\mu_{e_p}(x)\, f_{\gamma\cup p}(x) \right) e_\gamma \right\|_n =$$

$$= \sqrt{\sum_{\gamma\in\Gamma(n)} \left( \sum_{p\notin\gamma} \beta^\mu_{e_p}(x)\, f_{\gamma\cup p}(x) \right)^2} \leqslant \sqrt{\sum_{\gamma\in\Gamma(n)} \left( \sum_{p\notin\gamma} (\beta^\mu_{e_p}(x))^2 \sum_{p\notin\gamma} f^2_{\gamma\cup p}(x) \right)} \leqslant$$

$$\leqslant \sqrt{\sum_{p=1}^{\infty}(\beta^{\mu}_{e_p}(x))^2 \sum_{\gamma\in\Gamma(n)}\sum_{p\notin\gamma} f^2_{\gamma\cup p}(x)} \leqslant \sqrt{n+1}\,\|\beta^{\mu}\|_{H} \sqrt{\sum_{\gamma_1\in\Gamma(n+1)} f^2_{\gamma_1}(x)} =$$
$$= \sqrt{n+1}\,\|\beta^{\mu}(x)\|_{H}\,\|f(x)\|_{n+1}\,. \quad (2.5)$$

From this and (2.4), it follows that $\|\beta^{\mu} \lrcorner f\|_{n,p} < \infty$. By (2.4), $\|\delta f\|_{n,p} < \infty$. Hence, $\|\beta^{\mu} \lrcorner f + \delta f\|_{n,p} < \infty$, i.e. $-\beta^{\mu} \lrcorner f - \delta f \in \Omega^n_p$. Next,

$$d\omega = \sum_{\gamma\in\Gamma(n),\, p\in\gamma} d_{e_p}\omega_\gamma\, e_p \wedge e_\gamma = \sum_{\gamma\in\Gamma(n), p\notin\gamma} (-1)^{k_p-1}\, d_{e_p}\omega_\gamma\, e_{\gamma\cup p} =$$
$$= \sum_{\gamma_1\in\Gamma(n+1)} \Big( \sum_{\substack{\gamma\subset\gamma_1,\\ p\in\gamma_1\setminus\gamma}} (-1)^{k_p-1}\, d_{e_p}\omega_\gamma \Big)\, e_{\gamma_1}\ ,$$

where $k_p$ is the number of $p$ in the sequence $\gamma_1$. Applying the integration by parts formula [5] (one can easily verify the conditions under which this formula holds), we get

$$\langle d\omega, f\rangle_n = \sum_{\gamma_1\in\Gamma(n+1)} \int_X f_{\gamma_1}(x)\Big( \sum_{\substack{\gamma\subset\gamma_1,\\ p\in\gamma_1\setminus\gamma}} (-1)^{k_p-1}\, d_{e_p}\omega_\gamma(x)\Big)\, \mu(dx) =$$
$$= \sum_{\substack{\gamma_1\in\Gamma(n+1),\\ \gamma\subset\gamma_1,\, p\in\gamma_1\setminus\gamma}} (-1)^{k_p-1} \int_X f_{\gamma_1}(x)\, d_{e_p}\omega_\gamma(x)\, \mu(dx) =$$
$$=- \sum_{\substack{\gamma_1\in\Gamma(n+1),\\ \gamma\subset\gamma_1,\, p\in\gamma_1\setminus\gamma}} (-1)^{k_p-1} \int_X \omega_\gamma(x)\, (d_{e_p} f_{\gamma_1}(x) + f_{\gamma_1}(x)\, \beta^{\mu}_{e_p}(x))\, \mu(dx) =$$
$$= - \int_X \sum_{\gamma\in\Gamma(n),\, p\notin\gamma} (-1)^{k_p-1}\omega_\gamma(x)\, (d_{e_p} f_{\gamma\cup p}(x) + f_{\gamma\cup p}(x)\, \beta^{\mu}_{e_p}(x))\, \mu(dx). \quad (2.6)$$

We have changed the order of summation and integration when passing to the last expression in (2.6) using the Lebesgue theorem, since the sequence of partial sums of the last expression under the integral sign in (2.6) is majorized by an integrable function. This follows from the definition of the spaces $A^{n+1}_p$, $B^n_q$, from the Cauchy-Bunyakovsky inequality, and from the inequality

$$\sqrt{\sum_{\gamma\in\Gamma(n)} \Big( \sum_{p\notin\gamma} \beta^{\mu}_{e_p}(x)\, f_{\gamma\cup p}(x) \Big)^2} \leqslant \sqrt{n+1}\, \|\beta^{\mu}(x)\|_{H}\, \|f(x)\|_{n+1}\ ,$$

which was proved together with the estimate (2.5). By the same argument, the absolute convergence of all series in (2.6) holds, and hence the order of summation these series can be chosen arbitrary. We rewrite the expressions for $\delta f$ and $\beta^\mu \lrcorner f$

$$\delta f(x) = \sum_{p\in\gamma_1\in\Gamma(n+1)} d_{e_p} f_{\gamma_1}(x)\, e_p \lrcorner e_{\gamma_1} = \sum_{\gamma\in\Gamma(n)} \left( \sum_{p\notin\gamma} (-1)^{k_p-1}\, d_{e_p} f_{\gamma\cup p}(x) \right) e_\gamma \ ,$$

$$\beta^\mu(x) \lrcorner f(x) = \sum_{p\in\gamma_1\in\Gamma(n+1)} f_\gamma(x)\, \beta^\mu_{e_p}(x)\, e_p \lrcorner e_{\gamma_1} =$$

$$= \sum_{\gamma\in\Gamma(n),\, p\notin\gamma} (-1)^{k_p-1} f_{\gamma\cup p}(x) \beta^\mu_{e_p}(x)\, e_\gamma \ .$$

This and (2.6) imply the statement of the theorem. □

## 2.2 The Stokes formula

As before, let $X$ be a locally convex space, $H$ be its vector subspace which is a Hilbert space relative to a scalar product $(\,\cdot\,,\,\cdot\,)$, $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $H$. We assume that $H$ is dense in $X$ and that the identical embedding of $H$ into $X$ is continuous. By $\Xi_n$ $(n \in N)$, we denote the vector space of all differential forms of degree $n$ differentiable along $H$. Let, in addition to this, the ranges of values of differential forms from $\Xi_n$ and their differentials be bounded. By $S_n$, we denote the space of all differential forms of codegree $n$ differentiable along the subspace $H$, which are Radon measures, and assume that the differential forms from $S_n$ and their differentials are measures of a bounded variation. By $\bar{S}_n$ and $\bar{\Xi}_n$ $(n \in \mathbb{N})$, we denote pseudo-topological vector spaces of all linear continuous functionals on $\Xi_n$ and $S_n$, respectively. We assume that $\bar{S}_n$ and $\bar{\Xi}_n$ contain $S_n$ and $\Xi_n$ as dense subsets. Let $V$ be a domain in the space $X$ such that its boundary $\partial V$ can be covered by a union of a finite number of surfaces $\mathcal{U}_i$ of codimension 1; by a surface we understand an object defined in the previous paragraph. We assume that the sets $\mathcal{U}_i$ covering $\partial V$, and all their intersections, possess property $(*)$ (stated [1], Theorem 9). In what follows, we will use the notations ($\nu^\varepsilon$, $n^{\partial V}$, $\nu^{\partial V}$, $H_x$, $\varepsilon_b$, $P_{\partial V}$, $B^\varepsilon$ for a Borel set $B \subset \partial V$), introduced in the section 1.2. Here, instead of the symbol $G$ which stays for the surface, we will use the symbol $\partial V$. Note that by Theorem 9, $\lim_{\varepsilon\to 0} \nu^\varepsilon(\partial V) = \nu^{\partial V}(\partial V)$. Further assume that the indicator $\mathbb{I}_V$ of the set $V$ is an element of the space $\bar{\Xi}_0$. In addition to that, we assume that the mapping $d : \Xi_0 \to \Xi_1$ can be extended to a continuous mapping $\bar{\Xi}_0$ $\bar{\Xi}_1$, and the mapping $d : S_n \to S_{n-1}$, to a continuous mapping $\bar{S}_n \to \bar{S}_{n-1}$. Further, for every measure $\nu \in S_0$, assume that the mapping $\Xi_1 \to S_1$, $f \mapsto f\nu$ can be extended to a continuous mapping $\bar{\Xi}_1$ $\bar{S}_1$, and for all $m$ and $n$, the mapping $\Xi_m \times \Xi_n \to \Xi_{m+n}$, $(f, g) \mapsto f \wedge g$ can be extended to a continuous mapping $\bar{\Xi}_m \times \bar{\Xi}_n \to \bar{\Xi}_{m+n}$, and the mapping

$\Xi_m \times S_n \to S_{n-m}$, $(f, \omega) \mapsto f \wedge \omega$, to a continuous mapping $\bar{\Xi}_m \times \bar{S}_n \to \bar{S}_{n-m}$ $(n \geqslant m)$. We denote the extended mapping by the same symbols. The operation of the functional $f \in \bar{S}_1$ on the element $g \in \Xi_1$, we denote by $\langle f, g \rangle$. Assume that each sequence of elements from $\Xi_0$ which converges pointwise to an element of the space $\bar{\Xi}_0$, converges to this element also in $\bar{\Xi}_0$.

THEOREM 13. *Let $\nu \in S_0$. Then $(d\,\mathbb{I}_V) \cdot \nu \, (\in \bar{S}_1)$ is a concentrated on $\partial V$ Radon measure on $X$, taking values in $H$, and the following inequality holds (considered for elements of $\bar{S}_1$)*

$$(d\,\mathbb{I}_V) \cdot \nu = -n^{\partial V} \cdot \nu^{\partial V}, \tag{2.7}$$

*where $\nu^{\partial V}$ denotes the surface measure on $\partial V$* [1]*, generated by the measure $\nu$.*

*Proof.* Let $h^\varepsilon : (-\varepsilon_b, \varepsilon_b) \to [0,1]$, $\varepsilon < \varepsilon_b$, be an infinitely many times differentiable function such that

$$h^\varepsilon(\tau) = \begin{cases} -\frac{\tau}{2\varepsilon} + \frac{1}{2}\,, & \text{if } \tau \in (-(\varepsilon - \varepsilon^2), \varepsilon - \varepsilon^2), \\ 1, & \text{if } \tau \in (-\varepsilon_b, -\varepsilon), \\ 0, & \text{if } \tau \in (\varepsilon, \varepsilon_b), \end{cases}$$

and such that on the intervals $(-\varepsilon, -(\varepsilon - \varepsilon^2))$ and $(\varepsilon - \varepsilon^2, \varepsilon)$, the absolute value of its derivative changes monotonically from 0 till $\frac{1}{2\varepsilon}$, and from $\frac{1}{2\varepsilon}$ till 0, respectively. For $\varepsilon < \varepsilon_b$, define the functions $f^\varepsilon : X \to \mathbb{R}$ in the following way:

$$f^\varepsilon(x) = \begin{cases} h^\varepsilon(\tau), & \text{if } x = y + \tau n^{\partial V}(y),\ y \in \partial V,\ \tau \in (-\varepsilon_b, \varepsilon_b), \\ 1, & \text{if } x \in V \diagdown (\partial V)^{\varepsilon_b}, \\ 0, & \text{if } x \notin V \cup (\partial V)^{\varepsilon_b}. \end{cases}$$

Calculate

$$d_{e_p} f^\varepsilon(x + \tau n^{\partial V}(x)) = \frac{d}{dt} f^\varepsilon(x + \tau n^{\partial V}(x) + t e_p)\bigg|_{t=0}$$

for $\tau \in (-(\varepsilon - \varepsilon^2), \varepsilon - \varepsilon^2)$ and $x \in \partial V$. If $t$ is sufficiently small, then there exist $x_t$ and $\tau_t$, such that

$$x + \tau n^{\partial V}(x) + t e_p = x_t + \tau_t n^{\partial V}(x_t). \tag{2.8}$$

Let $n_p^{\partial V}$ be coordinate of the vector $n^{\partial V}$ in the basis $\{e_p\}$. Transforming (2.8) and multiplying the both parts by $n^{\partial V}(x)$, we get

$$(x - x_t, n^{\partial V}(x)) + \tau + t n_p^{\partial V}(x) = \tau_t (n^{\partial V}(x), n^{\partial V}(x_t)),$$

[1] in the sense of [35]

and hence,

$$\tau_t = \frac{tn_p^{\partial V}(x)}{(n^{\partial V}(x), n^{\partial V}(x_t))} + \frac{\tau + (x - x_t, n^{\partial V}(x))}{(n^{\partial V}(x), n^{\partial V}(x_t))} \ . \tag{2.9}$$

Note that $x_t = P_{\partial V}(x + \tau n^{\partial V}(x) + te_p)$. Analogously to how it was done at the proof of Lemma 6 in [1], it can be proved that the derivative $\frac{d}{dt} x_t\big|_{t=0}$ exists in the topology of the space $H$. From the results of [1] (Lemmas 1 and 2), it follows that the derivative $\frac{d}{dt} n^{\partial V}(x_t)\big|_{t=0}$ exists in the topology of the space $H$ as well. Taking into account this, we show that $\frac{d}{dt}(n^{\partial V}(x), n^{\partial V}(x_t))\big|_{t=0} = 0$ $\frac{d}{dt}(x - x_t, n^{\partial V}(x))\big|_{t=0} = 0$. The last fact is obvious, since $\frac{d}{dt} x_t\big|_{t=0} \in H_x$, and $n^{\partial V}(x)$ is orthogonal to $H_x$. Further we have

$$0 = \frac{d}{dt} \|n^{\partial V}(x_t)\|^2 \bigg|_{t=0} = 2\Big(n^{\partial V}(x), \frac{d}{dt} n^{\partial V}(x_t)\bigg|_{t=0}\Big) = $$
$$= 2\,\frac{d}{dt}(n^{\partial V}(x), n^{\partial V}(x_t))\bigg|_{t=0} .$$

From this and from (2.9), it follows that $\frac{d}{dt}\tau_t\big|_{t=0} = n_p^{\partial V}(x)$. Further, taking into account that $f^\varepsilon(x + \tau n^{\partial V}(x) + te_p) = -\frac{\tau_t}{2\varepsilon} + \frac{1}{2}$, we get $d_{e_p} f^\varepsilon(x + \tau n^{\partial V}(x)) = -\frac{n_p^{\partial V}(x)}{2\varepsilon}$, and hence,

$$df^\varepsilon(x + \tau n^{\partial V}(x)) = -\frac{n^{\partial V}(x)}{2\varepsilon} \ . \tag{2.10}$$

For $\tau$ which belong to one of the interval $(-\varepsilon, -(\varepsilon - \varepsilon^2))$ or $(\varepsilon - \varepsilon^2, \varepsilon)$, the differential $df^\varepsilon(x + \tau n^{\partial V}(x))$ is calculated in the same way as before: $f^\varepsilon(x + \tau n^{\partial V}(x) + te_p) = h^\varepsilon(\tau_t)$

$$df^\varepsilon(x + \tau n^{\partial V}(x)) = (h^\varepsilon)'(\tau) n^{\partial V}(x) \ , \tag{2.11}$$

From this, by the definition of the function $h^\varepsilon$, it follows that

$$\|df^\varepsilon(x)\| < \frac{1}{2\varepsilon} \quad \text{for all } x \in X. \tag{2.12}$$

Let $g \in \Xi_1$. We have

$$\langle (d\,\mathbb{I}_V)\cdot \nu, g\rangle = \lim_{\varepsilon\to 0} \int_{(\partial V)^\varepsilon} (df_\varepsilon(x), g(x))\nu(dx) \ .$$

Let $x \in \partial V$. Then $(n^{\partial V}(x), g(x + tn^{\partial V}(x)))$ is a differentiable in $t$ function from $[0, \varepsilon_b)$ to $\mathbb{R}$. By assumption, $g$ has a bounded derivative along the subspace $H$, say by a constant $M$. For all $x \in \partial V$, $|t| < \varepsilon$ we have

$$|(n^{\partial V}(x), g(x + tn^{\partial V}(x))) - (n^{\partial V}(x), g(x))| \leqslant$$
$$\leqslant \bigg|\Big(n^{\partial V}(x), \frac{d}{dt} g(x + tn^{\partial V}(x))\bigg|_{t=t_0}\Big)\bigg| \cdot t \leqslant \|g'(x + t_0 n^{\partial V}(x)) n^{\partial V}(x)\| \cdot t \leqslant$$
$$\leqslant \|g'(x + t_0 n^{\partial V}(x))\|_2 \cdot t < M\varepsilon,$$

where $t_0 < t$. Define a function $\tilde{g} : (\partial V)^{\varepsilon_b} \to H$ in the following way: for $x \in \partial V$, $|t| < \varepsilon_b$, set $\tilde{g}(x + tn^{\partial V}(x)) = g(x)$. Then, taking into account the last sequence of inequalities, formulas (2.10), (2.11), and the definition of the function $h^\varepsilon$, for all $x \in (\partial V)^\varepsilon$,

$$|(df^\varepsilon(x), g(x)) - (df^\varepsilon(x), \tilde{g}(x))| < \frac{M}{2} .$$

From this, it follows that

$$\begin{aligned}
\langle (d\,\mathbb{I}_V)\cdot \nu, g\rangle &= \lim_{\varepsilon\to 0}\int_{(\partial V)^\varepsilon} (df_\varepsilon(x), \tilde{g}(x))\nu(dx) = \\
&= -\lim_{\varepsilon\to 0}\int_{(\partial V)^\varepsilon} \frac{1}{2\varepsilon}\,(n^{\partial V}(P_{\partial V}x), g(P_{\partial V}x))\nu(dx) + \\
&\quad + \lim_{\varepsilon\to 0}\int_{(\partial V)^\varepsilon \setminus (\partial V)^{\varepsilon-\varepsilon^2}} (df_\varepsilon(x) + \frac{1}{2\varepsilon}\,n^{\partial V}(P_{\partial V}x), \tilde{g}(x))\nu(dx) = \\
&= -\lim_{\varepsilon\to 0}\int_{\partial V} (n^{\partial V}(x), g(x))\nu^\varepsilon(dx) ,
\end{aligned} \tag{2.13}$$

where the measure $\nu^\varepsilon$ was introduced in the previous section. For the second integral in the expression next to the last, by (4), and Theorem (9) [2] and taking into account that the function $g$ is bounded by a constant $M$, we obtain that

$$\begin{aligned}
&\Big|\int_{(\partial V)^\varepsilon \setminus (\partial V)^{\varepsilon-\varepsilon^2}} (df_\varepsilon(x) - \frac{1}{2\varepsilon} n^{\partial V}(P_{\partial V}x), \tilde{g}(x))\nu(dx)\Big| \leqslant \\
&\quad\leqslant M\,\frac{\nu((\partial V)^\varepsilon \setminus (\partial V)^{\varepsilon-\varepsilon^2})}{\varepsilon} = 2\,\Big(\frac{\nu((\partial V)^\varepsilon)}{2\varepsilon} - \frac{\nu((\partial V)^{\varepsilon-\varepsilon^2})}{2(\varepsilon-\varepsilon^2)}\frac{\varepsilon-\varepsilon^2}{\varepsilon}\Big) \to 0 ,
\end{aligned}$$

as $\varepsilon \to 0$. Further, we fix an arbitrary $\sigma > 0$, and let, $\sigma' = \frac{\sigma}{2(M+\nu^{\partial V}(\partial V))}$. Since $\nu^{\partial V}$ is a Radon measure (see. [35]), then there exists a compact $K_\sigma \subset \partial V$, such that $\nu^{\partial V}(\partial V \setminus K_\sigma) < \sigma'$. For each point $x_0 \in K_\sigma$ we fix a neighborhood $U_{x_0}$, which is contained in one of $\mathcal{U}_i$, possesses property $(*)$ stated in the previous section, and such that for the function $\varphi(x) = (n^{\partial V}(x), g(x))$, the inequality $|\varphi(x)-\varphi(x_0)| < \sigma'$ holds for all $x \in U_{x_0}$. We choose a finite number of neighborhoods $U_x$, $x \in K_\sigma$, (let it be neighborhoods $U_i$ of the points $x_i$), and denote their union by $O_\sigma$. It is clear that $\nu^{\partial V}(\partial V \setminus O_\sigma) < \sigma'$, and by construction of $O_\sigma$, there exists the limit $\lim_{\varepsilon\to 0}\nu^\varepsilon(O_\sigma) = \nu^{\partial V}(O_\sigma)$, hence the limit $\lim_{\varepsilon\to 0}\nu^\varepsilon(\partial V \setminus O_\sigma) = \nu^{\partial V}(\partial V \setminus O_\sigma)$ exists too. Further let $B_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j$, and $\varphi_\sigma : O_\sigma \to \mathbb{R}$ is such that

[2]In the previous section the measure $\nu$ was supposed to be non-negative. However, Theorem (9) which we apply in this proof, holds without the assumption on non-negativity of the measure $\nu$ (without changes in its proof).

$\varphi_\sigma = \sum_i \varphi(x_i)\,\mathbb{I}_{B_i}$. It is clear that on $O_\sigma$, $|\varphi(x) - \varphi_\sigma(x)| < \sigma'$. We have

$$\lim_{\varepsilon\to 0}\int_{\partial V}\varphi(x)\nu^\varepsilon(dx) = \lim_{\varepsilon\to 0}\int_{O_\sigma}\varphi_\sigma(x)\nu^\varepsilon(dx)+ \\ +\lim_{\varepsilon\to 0}\Big(\int_{O_\sigma}(\varphi(x)-\varphi_\sigma(x))\nu^\varepsilon(dx) + \int_{\partial V\setminus O_\sigma}\varphi(x)\nu^\varepsilon(dx)\Big). \quad (2.14)$$

By the definition of $\varphi_\sigma$ $\int_{O_\sigma}\varphi_\sigma(x)\nu^\varepsilon(dx) = \sum_i \varphi(x_i)\nu^\varepsilon(B_i)$, where the sum contains a finite number of terms, and for the sets $B_i$, the equality $\lim_{\varepsilon\to 0}\nu^\varepsilon(B_i) = \nu^{\partial V}(B_i)$ holds by their construction and by Theorem (9). Hence $\lim_{\varepsilon\to 0}\int_{O_\sigma}\varphi_\sigma(x)\nu^\varepsilon(dx) = \int_{O_\sigma}\varphi_\sigma(x)\nu^{\partial V}(dx)$. The limit in the second tern in (2.14) exists by the existence of the two other limits. By what we just proved, we continue (2.14).

$$\lim_{\varepsilon\to 0}\int_{\partial V}\varphi(x)\nu^\varepsilon(dx) = \int_{\partial V}\varphi(x)\nu^{\partial V}(dx)- \\ -\int_{\partial V\setminus O_\sigma}\varphi(x)\nu^{\partial V}(dx) - \int_{O_\sigma}(\varphi(x)-\varphi_\sigma(x))\nu^{\partial V}(dx)+ \\ +\lim_{\varepsilon\to 0}\Big(\int_{O_\sigma}(\varphi(x)-\varphi_\sigma(x))\nu^\varepsilon(dx) + \int_{\partial V\setminus O_\sigma}\varphi(x)\nu^\varepsilon(dx)\Big).$$

Estimate the last three terms. We have

$$\Big|\int_{O_\sigma}(\varphi(x)-\varphi_\sigma(x))\nu^\varepsilon(dx) + \int_{\partial V\setminus O_\sigma}\varphi(x)\nu^\varepsilon(dx)\Big| \leqslant \sigma'\nu^\varepsilon(O_\sigma) + M\nu^\varepsilon(\partial V\setminus O_\sigma).$$

As $\varepsilon \to 0$, passing to the limit on both parts of this inequality, and taking into account that the limit on the left-hand side exists by the fact of the existence of the limit under the operation of taking of the absolute value, we obtain

$$\Big|\lim_{\varepsilon\to 0}\Big(\int_{O_\sigma}(\varphi(x)-\varphi_\sigma(x))\nu^\varepsilon(dx) + \int_{\partial V\setminus O_\sigma}\varphi(x)\nu^\varepsilon(dx)\Big)\Big| \leqslant \sigma'\nu^{\partial V}(\partial V) + M\sigma' .$$

Analogously, we prove the two other estimates

$$\Big|\int_{\partial V\setminus O_\sigma}\varphi(x)\nu^{\partial V}(dx)\Big| < M\sigma', \qquad \Big|\int_{O_\sigma}(\varphi(x)-\varphi_\sigma(x))\nu^{\partial V}(dx)\Big| < \sigma'\nu^{\partial V}(\partial V).$$

From this, it follows that

$$\Big|\lim_{\varepsilon\to 0}\int_{\partial V}\varphi(x)\nu^\varepsilon(dx) - \int_{\partial V}\varphi(x)\nu^{\partial V}(dx)\Big| < \sigma,$$

which, by arbitrary choice of $\sigma$, means

$$\lim_{\varepsilon \to 0} \int_{\partial V} (n^{\partial V}(x), g(x)) \nu^{\varepsilon}(dx) = \int_{\partial V} (n^{\partial V}(x), g(x)) \nu^{\partial V}(dx) \ ,$$

and, together with (2.13), implies

$$\langle (d\,\mathbb{I}_V) \cdot \nu, g \rangle = - \int_{\partial V} (n^{\partial V}(x), g(x)) \nu^{\partial V}(dx) \ ,$$

which is equivalent to (2.7). The theorem is proved. □

By the assumptions on continuous extension of the mappings $d$ and $\wedge$ on differential forms from $\bar{\Xi}_m$, equality (2.1) holds for differential forms $g$ from $\bar{\Xi}_m$. Further, by Theorem 13, for elements $\omega \in S_1$, the following equality holds:

$$d\,\mathbb{I}_V \wedge \omega = -\, n^{\partial V} \wedge \omega^{\partial V}, \tag{2.15}$$

where $\omega^{\partial V} \in S_1$ is such that its every component $\omega_p^{\partial V}$ is a surface measure generated by the measure $\omega_p$.

Definition 14. *We define the integral of a differential form $\omega \in S_1$ over the surface $\partial V$ in the following way:*

$$\int_{\partial V} \omega = \int_{\partial V} (n^{\partial V}, \omega^{\partial V}(dx)).$$

Note that equality (2.15) implies $\int_{\partial V} \omega = -(d\,\mathbb{I}_V \wedge \omega)(X)$.

Theorem 14 (The Stockes formula). *For all $\omega \in S_1$, the formula*

$$\int_{\partial V} \omega = \int_V d\omega$$

*holds.*

*Proof.* We have

$$0 = d(\mathbb{I}_V \wedge \omega)(X) = (d\,\mathbb{I}_V \wedge \omega)(X) + (\mathbb{I}_V \wedge d\omega)(X),$$

and hence,

$$\int_V d\omega = (\mathbb{I}_V \wedge d\omega)(X) = -(d\,\mathbb{I}_V \wedge \omega)(X) = \int_{\partial V} \omega.$$

The theorem is proved. □

Let $V_1, \ldots, V_n$ be domains in $X$, possessing the same properties as the domain $V$, such that for each $i$, the surfaces $\partial V \cap \partial V_1 \cap \ldots \cap \partial V_i$ and $\partial V_1 \cap \ldots \cap \partial V_i$ can be covered with a finite number of $i+1$ and $i$, respectively (by a surface of codimension $n$, we understand the object defined as in (5)). Let $G = V \cap \partial V_1 \cap \ldots \cap \partial V_n$, $\partial G = \partial V \cap \partial V_1 \cap \ldots \cap \partial V_n$, i.e. $G$ is the domain in $\partial V_1 \cap \ldots \cap \partial V_n$, with the boundary $\partial G$. We construct a system of normal vectors to $\partial G$ at the point $x \in \partial G$. Choose a vector $n^{\partial V \cap \partial V_1}(x)$, such that it belongs to the tangent (in $H$) space to $\partial V$ at the point $x$, and is orthogonal in $H$ to the tangent space to $\partial V \cap \partial V_1$ at the point $x$. The vector $n^{\partial V \cap \partial V_1 \cap \ldots \cap \partial V_i}(x)$, $1 < i \leqslant n$, we choose so, that it belongs to the tangent space to the surface $\partial V \cap \partial V_1 \cap \ldots \cap \partial V_{i-1}$ at the point $x$, and is orthogonal to the tangent space to the surface $\partial V \cap \partial V_1 \cap \ldots \cap \partial V_i$ at the point $x$. Obviously, we can choose all of these vectors, since the tangent space to $\partial V \cap \partial V_1 \cap \ldots \cap \partial V_i$ at the point $x$ considered as a subspace of the tangent space to $\partial V \cap \partial V_1 \cap \ldots \cap \partial V_{i-1}$ at the point $x$, has the codimension 1. We choose the vector $n^{\partial V \cap \partial V_1 \cap \ldots \cap \partial V_i}$ so, that it is directed inside the domain $V_i$, if applied at the point $x$.

Definition 15. *Let $\omega \in S_n$. We define the integral of a differential form $\omega$ over a surface of the form $\Gamma = \partial V_1 \cap \ldots \cap \partial V_n$, as follows:*

$$\int_\Gamma \omega = \int_\Gamma (n^{\partial V_1}(x) \wedge n^{\partial V_1 \cap \partial V_2}(x) \wedge \ldots \wedge n^{\partial V_1 \cap \cdots \cap \partial V_n}(x), \omega^\Gamma(dx))_n \ ,$$

*where $\omega^\Gamma = \sum_{\gamma \in \Gamma_n} \omega^\Gamma_\gamma e_\gamma$, and $\omega^\Gamma_\gamma$ is the surface measure on $\Gamma$ generated by the measure $\omega_\gamma$.*

# Chapter 3

# Brownian sheet with values in a compact Riemannian manifold

In this section, we construct a two times parameter field $\mathbf{W}_M^x$ with values in a compact Riemannian manifold without boundary as a limit of stepwise conditioned Brownian sheets. Let $M$ be a $d$-dimensional compact Riemannian manifold without boundary isometrically embedded into $\mathbb{R}^m$, $C([0,1],M)$ the set of continuous functions on the interval $[0,1]$ taking values in $M$, $x$ a point on the manifold $M$. By a Brownian sheet with values in $\mathbb{R}^m$, we understand a family of $m$ independent standard Brownian sheets. The field $\mathbf{W}_M^x$ can be considered as a process with values in $C([0,1],M)$. We will show that the transition density at the time $t$ of this process is the distribution of a multiple Brownian motion on the manifold $M$. A multiple Brownian motion on a manifold with parameter $s$ is understood to be the continuous Markov process with the transition semigroup generated by the operator $-\frac{s}{2}\Delta_M$, where $\Delta_M$ is the Laplace operator on the manifold $M$.

## 3.1 The first step of construction of $\mathbf{W}_M^x$

Let $\mathbf{W}_{t,s}$ be an $n$-dimensional Brownian sheet. Consider $\mathbf{W}_{t,s}$ as a Brownian motion taking values in the space $C([0,1],\mathbb{R}^m)$ (in the sense of [18])[1]. We will denote this process by $\mathbf{W}_t$. Let us introduce the following notations: if $E$ is a locally convex space (LCS), then $E^t$ denotes $C([0,t],E)$; if $y \in C([0,1],\mathbb{R}^m)$ is a function of variable $s$ then $\mathbb{W}^y$ denotes the distribution of the process $\mathbf{W}_t^y = y + \mathbf{W}_t$. If $\psi \in C([0,1],\mathbb{R}^m)$, then we can consider the process $(\mathbf{W}_\psi^y)_t = \psi(t) + \mathbf{W}_t^y$.

[1]The Brownian motion with values in $C([0,1],G)$ where $G$ is a compact Lie group was constructed in [18] by Malliavin.

Let $\tilde{\mathbb{W}}^y_\psi$ be the distribution of this process, and $\mathbb{E}_{y,\psi}$ be the expectation with respect to $\tilde{\mathbb{W}}^y_\psi$. $U_\varepsilon(M)$ denotes the $\varepsilon$-neighborhood of $M$. We will consider $\mathbf{W}^y_\psi$ for the functions $y$ and $\psi$ with $y(0) \in M, \psi(0) = 0$. Our goal in this section is to prove the existence of the following limit relative to the family of bounded continuous cylinder functions. This limit defines a measure $\tilde{\mathbb{W}}^y_{M,\psi,s,t}$:

$$\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\tilde{\mathbb{W}}^y_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{y,\psi}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^y_\psi)_t(s)\in U_\varepsilon(M)\}}\}}{\tilde{\mathbb{W}}^y_\psi\{(\mathbf{W}^y_\psi)_t(s)\in U_\varepsilon(M)\}}\,. \tag{3.1}$$

Before we prove the existence of this limit consider the process $(\mathbf{W}^z_{\psi,s})_t = \psi(t) + B^s_t$, where $\psi : [0,1] \to \mathbb{R}^m$ is a continuous function with $\psi(0) = 0$, and $B^s_t$ is a multiple Brownian motion with the parameter $s$ starting at $z$. The results obtained for this process will be applied in the following construction.

*Some results related to the process* $\mathbf{W}^{s,\psi,z}_t$. Let $\mathbb{W}^z_{\psi,s}$ denote the distribution of the process $(\mathbf{W}^z_{\psi,s})_t$, $\mathbb{E}_{z,\psi,s}$ the expectation relative to this distribution. We will prove the existence of the week limit of the measures on the right-hand side relative to the family of cylinder functions. More precisely we will prove that

$$\int\limits_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{z,\psi,s}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^z_{\psi,s})_t\in U_\varepsilon(M)\}}\}}{\mathbb{W}^z_{\psi,s}\{(\mathbf{W}^z_{\psi,s})_t\in U_\varepsilon(M)\}}\,. \tag{3.2}$$

Find a function $\tilde{f} : \mathbb{R}^{k+1} \to \mathbb{R}$ and a finite number of points $\tau_1, \ldots, \tau_k$ so that $f(\omega) = \tilde{f}(\omega(\tau_1), \ldots, \omega(\tau_k), \omega(t))$.

$$\begin{aligned}
&\int\limits_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}\,\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega:\omega(t)\in U_\varepsilon(M)\}} \\
&= \lim_{\varepsilon\to 0} \frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega+z+\psi)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M-z-\psi(t))\}}\,\mathbb{W}^0_{0,s}(d\omega)}{\mathbb{W}^0_{0,s}\{\omega:\omega(t)\in U_\varepsilon(M-z-\psi(t))\}} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t)))}\cdot \int_{\mathbb{R}^m} \mathbb{P}^{\mathbb{W}}(\tau_1,0,dx_1)\int_{\mathbb{R}^m} \mathbb{P}^{\mathbb{W}}(\tau_2-\tau_1,x_1,dx_2)\ldots \\
&\qquad \int_{U_\varepsilon(M-\psi(t)-z)} \mathbb{P}^{\mathbb{W}}(t-\tau_k,x_k,dx_{k+1})\tilde{f}(x_1+z+\psi(\tau_1),\ldots,x_k+z+\psi(\tau_k), \\
&\qquad\qquad x_{k+1}+z+\psi(t))\,,
\end{aligned}$$

where $\mathbb{P}^{\mathbb{W}}(\tau, x, dz) = \frac{1}{(2\pi s\tau)^{\frac{m}{2}}} e^{\frac{|z-x|^2}{2s\tau}} dz$ the distribution of a multiple Brownian motion with the parameter $s$. Since the function under the integral sign is bounded, by Lebesgue theorem, it suffices to prove the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\int\limits_{(U_\varepsilon(M-\psi(t)-z))}\tilde f(x_1+z+\psi(\tau_1),\dots,x_{k+1}+z+\psi(t))\mathbb{P}^{\mathbb{W}}(t-\tau_k,x_k,dx_{k+1})}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t))}$$

$$=\lim_{\varepsilon\to 0}\frac{\int\limits_{U_\varepsilon(M-\psi(t)-z-x_k)}\tilde f(x_1+z+\psi(\tau_1),\dots,x_{k+1}+x_k+z+\psi(t))\mathbb{P}^{\mathbb{W}}(t-\tau_k,0,dx_{k+1})}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t))}.$$

Since the integration is with respect to $x_{k+1}$, the other variables are fixed and we may not take them into account when introducing the following notations: $g(x_{k+1})=\tilde f(x_1+z+\psi(\tau_1),\dots,x_{k+1}+x_k+z+\psi(t))$, $M_1=M-\psi(t)-z-x_k$, $M_2=M-z-\psi(t)$. Then we can rewrite the above limit as follows:

$$\lim_{\varepsilon\to 0}\frac{\frac{1}{(2\pi s(t-\tau_k))^{\frac{m}{2}}}\int_{U_\varepsilon(M_1)}e^{\frac{|x_k-x_{k+1}|^2}{2s(t-\tau_k)}}g(x_{k+1})dx_{k+1}}{\frac{1}{(2\pi st)^{\frac{m}{2}}}\int_{U_\varepsilon(M_2)}e^{\frac{|x_{k+1}|^2}{2st}}dx_{k+1}}\ . \tag{3.3}$$

Let $\lambda_\varepsilon=\frac{1}{vol_{m-d}(\varepsilon)}\,l|_{U_\varepsilon(M_1)}$ and $\mu_\varepsilon=\frac{1}{vol_{m-d}(\varepsilon)}\,l|_{U_\varepsilon(M_2)}$, where $l$ is the Lebesgue measure on $\mathbb{R}^m$. Then for the last limit we get

$$\lim_{\varepsilon\to 0}\frac{\int_{\mathbb{R}^m}g(x_{k+1})\,e^{-\frac{|x_{k+1}-x_k|^2}{2s(t-\tau_k)}}\lambda_\varepsilon(dx_{k+1})}{\int_{\mathbb{R}^m}e^{-\frac{|x_{k+1}|^2}{2st}}\mu_\varepsilon(dx_{k+1})}\ .$$

Lemma 19. *The measures $\lambda_\varepsilon$ and $\mu_\varepsilon$ converge weekly to the surface measures on $M_1$ and $M_2$ respectively as $\varepsilon$ goes to zero.*

*Proof.* Since $M$ is a smooth compact Riemannian manifold (and so are $M_1$ and $M_2$) we can find an $\varepsilon$-neighborhood of $M$ where the normal spaces at every point of $M$ do not have common points. Let $g$ be an arbitrary uniformly continuous bounded function $\mathbb{R}^m\to\mathbb{R}$. Choose the above $\varepsilon$ so that $|g(x)-g(y)|<\varepsilon$ for all $|x-y|<\varepsilon$. Further, denote by $\lambda^\varepsilon$ and $\mu^\varepsilon$ (with $\varepsilon$ on the top) the measures on $M_1$ and $M_2$ respectively defined on the Borel subsets of $M_1$ (respectively $M_2$) as $\lambda^\varepsilon(B)=\lambda_\varepsilon(B^\varepsilon)$, where $B^\varepsilon=\{x+tn_x,x\in B,n_x\in N(x),|n_x|=1,|t|\leqslant\varepsilon\}$ and $N(x)$ denotes the normal space at the point $x$ (and analogously for $\mu_\varepsilon$ and $M_2$). We prove the result only for the measure $\lambda_\varepsilon$ omitting the index 1 of $M_1$ for simplicity of notations. Clearly, each point of the $\varepsilon$-neighborhood where the normal spaces do not intersect each other, can be uniquely presented as $x+tn_x$ where $n_x\in N(x)$. Define a new function $\bar g(x+tn_x)=g(x)$ for $x\in M,|t|<\varepsilon$. Obviously, for all $x\in U_\varepsilon(M)$, $|\bar g(x)-g(x)|<\varepsilon$. As $\varepsilon$ goes to zero $\lambda^\varepsilon$ converges to the surface measure $\lambda_M$ on each Borel subset of $M$, and hence weakly. We have

$$\int_{U_\varepsilon(M)}\bar g(x)\lambda_\varepsilon(dx)=\int_M g(x)\lambda^\varepsilon(dx)\to\int_M g(x)\lambda_M(dx)\ .$$

Note that

$$\Big|\int_{\mathbb{R}^m} g(x)\lambda_\varepsilon(dx) - \int_{U_\varepsilon(M)} \bar{g}(x)\lambda_\varepsilon(dx)\Big| < \varepsilon\lambda_\varepsilon(U_\varepsilon(M)) < K\varepsilon$$

where $K$ is a constant. This implies

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^m} g(x)\lambda_\varepsilon(dx) = \int_M g(x)\lambda_M(dx)\ .$$

□

*Proof of the existence of the limit* (3.1). There exist $\tilde{f} : C([0,s],\mathbb{R}^m)^{k+1} \to \mathbb{R}$ and a finite number of points $\tau_1, \tau_2, \ldots, \tau_k$ such that $f(\omega) = \tilde{f}(\omega(\tau_1), \omega(\tau_2), \ldots, \omega(\tau_k), \omega(t))$, for which in turn there are a function $\tilde{\tilde{f}} : \mathbb{R}^{(k+1)\cdot(l+1)} \to \mathbb{R}$ and a finite number of points $\xi_1, \xi_2, \ldots, \xi_l$ so that $\tilde{f}(\omega(\tau_1), \omega(\tau_2), \ldots, \omega(\tau_k), \omega(t)) = \tilde{\tilde{f}}(\omega(\tau_1,\xi_1), \ldots, \omega(\tau_1,\xi_l), \ldots, \omega(t,\xi_l), \omega(t,s))$ (without loss of generality we can consider that the functions $\tilde{f}$ and $\tilde{\tilde{f}}$ depend on $\omega$ also at the points $t$ and $s$). Taking into account that $\mathbf{W}_t$ is a Markov process (see [18]), we get

$$\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\tilde{\mathbb{W}}^y_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0}\frac{\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t,s)\in U_\varepsilon(M)\}}\,\tilde{\mathbb{W}}^y_\psi(d\omega)}{\tilde{\mathbb{W}}^y_\psi\{\omega : \omega(t,s)\in U_\varepsilon(M)\}}$$

$$= \lim_{\varepsilon\to 0}\frac{\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega+y+\psi)\mathbb{I}_{\{\omega:\,\omega(t,s)\in U_\varepsilon(M-y(s)-\psi(t))\}}\tilde{\mathbb{W}}^0(d\omega)}{\tilde{\mathbb{W}}^0\{\omega : \omega(t,s)\in U_\varepsilon(M-y(s)-\psi(t))\}}$$

$$= \lim_{\varepsilon\to 0}\frac{\int\limits_{C([0,s],\mathbb{R}^m)}\mathbb{P}^{\tilde{\mathbb{W}}}(\tau_1,0,dw_1)\quad\int\limits_{C([0,s],\mathbb{R}^m)}\mathbb{P}^{\tilde{\mathbb{W}}}(\tau_2-\tau_1,w_1,dw_2)}{\tilde{\mathbb{W}}^0\circ\pi_s^{-1}\circ\pi_t^{-1}(U_\varepsilon(M-y(s)-\psi(t)))}\cdots$$

$$\int_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)))}\mathbb{P}^{\tilde{\mathbb{W}}}(t-\tau_k,w_k,dw_{k+1})$$

$$\tilde{f}\big(w_1+y+\psi(\tau_1),\ldots,w_k+y+\psi(\tau_k),w_{k+1}+y+\psi(t)\big)$$

Since the function under the integral sign is bounded it is suffices to prove the existence of the limit when $\varepsilon$ goes to 0 for the following quantity

$$\frac{\int\limits_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)))}\tilde{f}(w_1+y+\psi(\tau_1),\ldots,w_{k+1}+y+\psi(t))\mathbb{P}^{\tilde{\mathbb{W}}}(t-\tau_k,w_k,dw_{k+1})}{\tilde{\mathbb{W}}^0\circ\pi_s^{-1}\circ\pi_t^{-1}(U_\varepsilon(M-y(s)-\psi(t)))}$$

$$= \frac{\int\limits_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)-w_k(s)))}\tilde{f}(w_1+y+\psi(\tau_1),\ldots,w_{k+1}+w_k+y+\psi(t))\mathbb{P}^{\tilde{\mathbb{W}}}(t-\tau_k,0,dw_{k+1})}{\tilde{\mathbb{W}}^0\circ\pi_s^{-1}\circ\pi_t^{-1}(U_\varepsilon(M-y(s)-\psi(t)))}$$

Since the integration variable in the last integral is the variable $w_{k+1}$, then for simplicity, we introduce a function $g$ expressing dependence from $w_{k+1}$ only. More precisely, the function $g$ is such that $\tilde{f}(w_1 + y + \psi(\tau_1), \dots, w_{k+1} + w_k + y + \psi(t)) = g(w_{k+1}(\xi_1), \dots, w_{k+1}(\xi_l), w_{k+1}(s))$. Proceeding with the above calculation, we get

$$\frac{l(\xi_i, \tau_k, s, t) \int\limits_{\mathbb{R}^m} e^{-\frac{|x_1|^2}{2\xi_1(t-\tau_k)}} dx_1 \dots \int\limits_{U_\varepsilon(M-\psi(t)-y(s)-w_k(s))} g(x_1, \dots, x_{l+1})\, e^{-\frac{|x_{l+1}-x_l|^2}{2(s-\xi_l)(t-\tau_k)}} dx_{l+1}}{\frac{1}{(2\pi st)^{\frac{m}{2}}} \int\limits_{U_\varepsilon(M-\psi(t)-y(s))} e^{-\frac{|x_1|^2}{2st}} dx_1}$$

where $l(\xi_i, \tau_k, s, t) = \frac{1}{(2\pi(t-\tau_k))^{\frac{(l+1)m}{2}}} \frac{1}{(\xi_1(\xi_2-\xi_1)\cdots(s-\xi_l))^{\frac{m}{2}}}$ . And now we need to prove the existence of the limit

$$\lim_{\varepsilon\to 0} \frac{\int_{\mathbb{R}^m} \mathbb{I}_{U_\varepsilon(M-\psi(t)-y(s)-w_k(s))}\, g(x_1, \dots, x_{l+1})\, e^{-\frac{|x_{l+1}-x_l|^2}{2(s-\xi_l)(t-\tau_k)}} dx_{l+1}}{\int_{\mathbb{R}^m} \mathbb{I}_{U_\varepsilon(M-\psi(t)-y(s))}\, e^{-\frac{|x_1|^2}{2st}} dx_1}$$

which can be proved in exactly the same way as the existence of the limit (3.3). Hence, the existence of the limit (3.1) is proved for the family of cylinder functions.

## 3.2 Chernoff's theorem for evolution families

Let $[S, T] \subset \mathbb{R}$ be an interval, $A_t$, $t \in [S, T]$, be the generators of strongly continuous semigroups on a Banach space $E$, so that the space $F = \cap_t D(A_t)$ is dense in $E$. We assume that for all $x \in F$, $\sup\limits_{S \leqslant t \leqslant T} \|A_t x\|_E < \infty$. In the space $F$, we introduce the norm

$$\|x\|_F = \|x\|_E + \sup_{t\in[S,T]} \|A_t x\|_E \ .$$

$F$ is a Banach space relative to this norm, and all the operators $A_t : F \to E$ are continuous. Indeed, let $x_n$ be a Cauchy sequence relative to the norm $\|\cdot\|_F$. Then $x_n$ is a Cauchy sequence relative to the norm $\|\cdot\|_E$, and, in addition to this, $A_t x_n$ is a Cauchy sequence for every $t$. Let $x \in E$ be the limit of $x_n$ in $E$, $y_t \in E$ be the limit of $A_t x_n$. The fact that $A_t$ are closed operators, implies that $x \in D(A_t)$ for every $t$, and $A_t x = y_t$. Hence, $x \in F$. It is well known that the semigroup $T_t(\tau)$ generated by $A_t$ generates satisfies

$$\|T_t(\tau)\|_E \leqslant M_t e^{\omega_t \tau} \ .$$

We will assume that the constants $M_t$ and $\omega_t$ are independent of $t$, i.e. there exist constants $M > 0$ and $\omega > 0$ such that

$$\|T_t(\tau)\|_{E} \leqslant M e^{\omega\tau} \; . \tag{3.4}$$

Consider the non-autonomous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = A_t u(t) \\ u(s) = x \end{cases} \tag{3.5}$$

for $t, s \in [S, T]$, $s \leqslant t$. Further we will assume that the Cauchy problem (3.5) is well-posed [13] (forward solvable) [2]. According to [13], in this case there exist a strongly continuous evolution family $U(t, s)$, $s, t \in [S, T], s \leqslant t$, solving the Cauchy problem (3.5). It is well known that the evolution family solving (3.5), has the form of the pointwise limit:

$$U_{F}(t, s) = \lim_{\max \Delta t_j \to 0} \exp\{(t - t_{n-1})A_{\xi_n}\} \ldots \exp\{(t_1 - s)A_{\xi_1}\} \; , \tag{3.6}$$

where $\{s = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_{n-1} \leqslant t_n = t\}$ is a partition of $[s, t]$, $\xi_i \in [t_{i-1}, t_i]$. Consider another non-autonomous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = -\bar{A}_t u(t) \\ u(r) = x \end{cases} \tag{3.7}$$

where $t \leqslant r$ and $\bar{A}_t = A_{S+T-t}$.

Lemma 20. *The Cauchy problem* (3.7) *is backward solvable if and only if the Cauchy problem* (3.5) *is forward solvable. The evolution family $U_{B}(s, t)$, $s \leqslant t$, solving the Cauchy problem* (3.7), *satisfies*

$$U_{B}(t_1, t_2) U_{B}(t_2, t_3) = U_{B}(t_1, t_3) \tag{3.8}$$

*for $t_1 \leqslant t_2 \leqslant t_3$.*

*Proof.* Note that the limit (3.6) exists if and only if the limit

$$U_{B}(t', s') = \lim_{\max \Delta t_j \to 0} \exp\{(t'_1 - s')\bar{A}_{\xi'_1}\} \ldots \exp\{(t' - t'_{n-1})\bar{A}_{\xi'_n}\}$$

exists with $s' = S + T - t$, $t' = S + T - s$, $\xi'_i = S + T - \xi_{n-i+1}$, $t'_i = S + T - t_{n-i}$ (both limits exist pointwise), and $U_{F}(t, s) = U_{B}(s', t')$. The fact that $U_{F}(t, s)$, $t \geqslant s$, solves the Cauchy problem (3.5), immediately implies that $U_{B}(t, r)$, $t \leqslant r$, solves the Cauchy problem (3.7). The property (3.8) obviously holds for $U_{B}^{\mathcal{P}}(s', t') = \exp\{(t'_1 - s')\bar{A}_{\xi'_1}\} \ldots \exp\{(t' - t'_{n-1})\bar{A}_{\xi'_n}\}$ where $\mathcal{P} = \{s' \leqslant t'_1 \leqslant \ldots \leqslant t'_{n-1} \leqslant t'\}$, and can be obtained for $U_{B}(s', t')$ by taking the limit as the mesh tends to zero. □

[2] There exist several sufficient conditions for well-posedness. The corresponding references there are, for example, in [13].

LEMMA 21. *Let $R(\lambda, A)$ be the resolvent of the operator $A$. Assume the functions $[S,T] \to E$, $\xi \mapsto A_\xi x$, are continuous for every $x \in F$. Then, as $n \to \infty$, the pointwise convergence (on the space $F$) of the Yosida approximants [13] $A_\xi^{(n)} = n^2 R(n, A_\xi) - nI$ [13] to $A_\xi$ is uniform in $\xi$.*

*Proof.* Show first that for $\lambda > \omega$,

$$\|(\lambda - \omega) R(\lambda, A_\xi)\| \leqslant M \ .$$

We use the following expression for the resolvent [13]

$$R(\lambda, A_\xi)x = \int_0^\infty e^{-\lambda\tau} T_\xi(\tau)\, x\, d\tau \ .$$

Using the inequality (3.4), we get

$$\begin{aligned} \|(\lambda - \omega) R(\lambda, A_\xi)x\|_E &\leqslant \int_0^\infty (\lambda - \omega) e^{-\lambda\tau} \|T_\xi(\tau)\| \|x\|_E \, d\tau \\ &\leqslant M \int_0^\infty (\lambda - \omega) e^{-(\lambda - \omega)\tau} d\tau \, \|x\|_E \leqslant M\, \|x\|_E \ . \end{aligned}$$

Further, according to [13], $[\omega, \infty) \subset \rho(A_\xi)$ for all $\xi$. Show that $nR(n, A)x \to x$, as $n \to \infty$, for all $x \in F$. For $n > \omega$, we get

$$\|nR(n, A_\xi)x - x\|_E = \|R(n, A_\xi) A_\xi x\| \leqslant \frac{M}{n - \omega}\, \|A_\xi x\|_E \leqslant \frac{M}{n - \omega} \|x\|_F \ .$$

Further, let $x \in F$, $z_\xi = A_\xi x$. Note that the set $\{\|A_\xi x\| : \xi \in [S,T]\}$ is a compact in $E$. For an arbitrary number $\varepsilon > 0$, find a finite $\varepsilon$-net $\{y_i\} \in F$ for this compact. Note that for sufficiently big $n$,

$$\|R(n, A_\xi)\| \leqslant \|(n - \omega) R(n, A_\xi)\| \leqslant M,$$

and hence,

$$\|nR(n, A_\xi)\| \leqslant \|(n - \omega) R(n, A_\xi)\| + \omega \|R(n, A_\xi)\| \leqslant (1 + \omega) M \ .$$

For every $z_\xi$, find $y_i$ so that $\|z_\xi - y_i\|_E < \varepsilon$. We have

$$\begin{aligned} \|nR(n, A_\xi) z_\xi - z_\xi\|_E &\leqslant \|nR(n, A_\xi) z_\xi - nR(n, A_\xi) y_i\|_E \\ &\quad + \|z_\xi - y_i\|_E + \|nR(n, A_\xi) y_i - y_i\|_E \\ &\leqslant M(1 + \omega)\, \|z_\xi - y_i\|_E + \|z_\xi - y_i\|_E \\ &\quad + \frac{M}{n - \omega} \max_i \|y_i\|_F. \end{aligned}$$

From this, we have

$$\overline{\lim_{n\to\infty}} \|nR(n, A_\xi)z_\xi - z_\xi\|_E \leqslant M(1+\omega)\varepsilon + \varepsilon \ ,$$

Since $\varepsilon$ was chosen arbitrary, this implies

$$\lim_{n\to\infty} \|nR(n, A_\xi)z_\xi - z_\xi\|_E = 0.$$

The lemma is proved. □

The stability of system of generators $A_t$ means that

$$\Big\|\prod_{i=1}^{n} e^{\theta_i A_{t_i}}\Big\| \leqslant Me^{\omega(\theta_1+\cdots+\theta_n)}$$

for some constants $M$ and $\omega$ [34]. Consider the system of generators $A'_t = A_t - \omega I$ [3]. This system is stable with stability constants $M$ and 0. This also implies that the semigroups $e^{\tau A'_t}$ are uniformly bounded.

LEMMA 22. *Let $A_t$, $t \in [S,T]$, be a stable system of generators of strongly continuous semigroups with the stability constants $M$ and $\omega = 0$, and the operators $A_t$ commute for different $t$. Further, let $\mathcal{P}_{t_1\ldots t_n}$ denote the totality of points $\{S \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant T\}$. Then, the norm*

$$|||x||| = \sup_{\substack{n,\theta_i\geqslant 0\\ \mathcal{P}_{t_1\ldots t_n}}} \|\prod_{i=1}^{n} e^{\theta_i A_{t_i}} x\|_E$$

*is equivalent to the original norm in $E$. In addition to this, each semigroup $T_t(\tau) = e^{\tau A_t}$ becomes a contraction semigroup relative to this norm.*

*Proof.* Let $x \in E$. By the definition of stable system of generators, we have

$$|||x||| \leqslant M\|x\|_E.$$

On the other hand

$$|||x||| = \sup_{\substack{n,\theta_i\geqslant 0\\ \mathcal{P}_{t_1\ldots t_n}}} \|\prod_{i=1}^{n} e^{\theta_i A_{t_i}} x\|_E \geqslant \|\prod_{i=1}^{n} e^{\theta_i A_{t_i}} x\|_E\Big|_{\theta_i=0\,\forall i} = \|x\|_E,$$

which implies the equivalence of $\|\cdot\|_E$ and $|||\cdot|||$. Further,

$$|||e^{\tau A_t}x||| = \sup_{\substack{n,\theta_i\geqslant 0\\ \mathcal{P}_{t_1\ldots t_n}}} \|e^{\tau A_t}\prod_{i=1}^{n} e^{\theta_i A_{t_i}} x\|_E \leqslant |||x|||.$$

□

[3]According to [13], $A'_t = A_t - \omega I$ generates the strongly continuous semigroup $e^{-\omega\tau}e^{\tau A_t}$.

In the sequel, we will assume that the norm $\|\!|\cdot|\!\|$ is constructed with the help of the system $\{A'_t\}$. Note that $D(A_t) = D(A'_t)$, and for the generators $A'_t$, all the assumptions of Lemma 21 are fulfilled. Hence, on the space $F$, the pointwise convergence of the Yosida approximants ${A'}_t^{(n)}$ to $A'_t$ is uniform in $t \in [S,T]$. This implies the uniform pointwise (on the space $F$) convergence of the (bounded) operators $A_t^{(n)} = {A'}_t^{(n)} + I\omega$ to the operators $A_t$.

Lemma 23. *Let $T_\xi^{(n)}(t)$ denote $e^{tA_\xi^{(n)}}$. Then, as $n \to \infty$, the convergence*

$$\frac{1}{t}\big(T_\xi(t) - T_\xi^{(n)}(t)\big)\, x \to 0 \tag{3.9}$$

*holds uniformly in $t$ and in $\xi$.*

*Proof.* By Lemma 22, we can prove the convergence (3.9) relative to the $\|\!|\cdot|\!\|$-norm. In this norm, all the operators $A'_\xi$ generate contraction semigroups. For the semigroups generated by Yosida approximants, we have $\|\!|e^{t{A'}_\xi^{(n)}}|\!\| \leqslant 1$ (see [13]). Then, $\|\!|T_\xi^{(n)}(t)|\!\| \leqslant e^{t\omega}$. The well-known identity gives

$$\big(T_\xi(t) - T_\xi^{(n)}(t)\big)x = \int_0^t T_\xi^{(n)}(t-s)T_\xi(s)\left(A_\xi x - A_\xi^{(n)} x\right) ds\ .$$

This implies

$$\|\!|T_\xi(t)x - T_\xi^{(n)}(t)x|\!\| \leqslant \int_0^t e^{(t-s)\omega} e^{s\omega} \|\!|A_\xi x - A_\xi^{(n)} x|\!\| ds \leqslant t e^{t\omega} \|\!|A_\xi x - A_\xi^{(n)} x|\!\|\ .$$

The lemma is proved □

Theorem 15. *Let $A_t$ be generators of strongly continuous semigroups, $B$ another generator of a strongly continuous semigroup. We assume that the following assumptions are fulfilled:*

1) *$G = F \bigcap D(B)$ is dense in $E$;*

2) *the functions $[S,T] \to E$, $t \mapsto A_t x$, are continuous for all $x \in F$;*

3) *$B$ commutes with every $A_t$;*

4) *$\{A_t\}$ is a stable system of generators with the stability constants $M$ and $\omega$;*

5) *the Cauchy problem (3.7) with the generators $A_t$ is well-posed (backward solvable) on $F$, $U(s,t)$ is the evolution family solving the Cauchy problem (3.7);*

6) *there exists a dense in $E$ set $D \subset G$ such that for every $s$ and $t$, $U(s,t)D \subset D$;*

7) *for each fixed $x \in D$, the families of functions $(s,r) \mapsto A_t U(s,r)\,x$, $(s,r) \mapsto BU(s,r)\,x$, are equicontinuous.*

*Let $Q_{t_1,t_2}$, $t_1, t_2 > 0$, be a two-parameter family of contractions on $E$ such that*

$$\frac{Q_{\tau,\tau+\Delta\tau} - I}{\Delta\tau}\, e^{aB} x \to A_\tau\, e^{aB} x \ , \tag{3.10}$$

*as $\Delta\tau \to 0$, for all $x \in D$, for all $a > 0$, and uniformly in $\tau$. Let $S \leqslant s < t \leqslant T$, $\{s = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of the interval $[s,t]$ so that $\max \Delta t_j \to 0$ as $n \to \infty$, where $\Delta t_j = t_{j+1} - t_j$ . Then, for all $x \in E$,*

$$Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n} x \to U(s,t)\,x \ ,$$

*as $n \to \infty$.*

*Proof.* The scheme of the proof is similar to which was used in [25]. In the space $G$, we introduce the norm:

$$\|x\|_{G} = \|x\|_{F} + \|Bx\|_{E} \ .$$

It could be easily shown (analogously to the case of the space $F$) that $G$ is a Banach space. Everywhere below, we will assume that the space $G$ is equipped with the $\|\cdot\|_{G}$-norm.

The operators $\Big(\frac{Q_{\tau,\tau+\Delta\tau}-I}{\Delta\tau} - A_\tau\Big)e^{aB} : G \to E$, are continuous. By Banach-Steinhaus theorem and the uniform (in $\tau$) pointwise convergence, these operators are uniformly bounded in $\tau$ and $\Delta\tau$. Let $x \in D$. By assumption (7), for every fixed $x \in D$ and every fixed $t$, the set $\{U(r,t)\,x, r \in [s,t]\}$ is a compact in $G$. This compactness along with the uniform boundedness of $\Big(\frac{Q_{\tau,\tau+\Delta\tau}-I}{\Delta\tau} - A_\tau\Big)e^{aB}$ and the convergence (3.10), imply the following uniform (in $\tau$) convergence

$$\lim_{\Delta\tau\to 0} \sup_{\tau,\, s\leqslant r\leqslant t} \Big\| \Big( \frac{Q_{\tau,\tau+\Delta\tau} - I}{\Delta\tau} - A_\tau \Big) e^{aB} U(r,t)\,x \Big\|_{E} = 0 \ . \tag{3.11}$$

Further, we have

$$Q_{t_0,t_1} Q_{t_1,t_2} \ldots Q_{t_{n-1},t_n} - U(s,t) = \sum_{j=0}^{n-1} Q_{t_1,t_2} \ldots Q_{t_{j-1},t_j} (Q_{t_j,t_{j+1}} - U(t_j,t_{j+1})) U(t_{j+1},t) \ .$$

Further, let $x \in D$. We get

$$
\begin{aligned}
&\|(Q_{t_0,t_1}Q_{t_1,t_2}\dots Q_{t_{n-1},t_n} - U(s,t))e^{aB}x\| \\
&\leqslant \sum_{j=0}^{n-1} \Delta t_j \Big\| \Big( \frac{Q_{t_j,t_{j+1}} - I}{\Delta t_j} - \frac{U(t_j,t_{j+1}) - I}{\Delta t_j} \Big) e^{aB} U(t_{j+1},t)\, x \Big\|_E \\
&\leqslant (t-s) \sup \Big\{ \Big\| \Big( \frac{Q_{\tau,\tau+\Delta\tau} - I}{\Delta\tau} - \frac{U(\tau,\tau+\Delta\tau) - I}{\Delta\tau} \Big) e^{aB} U(\tau+\Delta\tau,t)\, x \Big\|_E : \\
&\qquad\qquad\qquad \tau \in [s,t), \Delta\tau < \max\{\max_j \Delta t_j, t-\tau\} \Big\} \\
&\leqslant (t-s) \sup \Big\{ \Big\| \Big( \frac{Q_{\tau,\tau+\Delta\tau} - I}{\Delta\tau} - A_\tau \Big) e^{aB} U(r,t)\, x \Big\|_E : \tau, r \in [s,t), \Delta\tau < \max_j \Delta t_j \Big\} \\
&+ (t-s) \sup \Big\{ \Big\| \frac{U(\tau,\tau+\Delta\tau) - e^{\Delta\tau A_\tau}}{\Delta\tau} e^{aB} U(r,t) x \Big\|_E : \tau, r \in [s,t), \Delta\tau < \max_j \Delta t_j \Big\} \\
&+ (t-s) \sup \Big\{ \Big\| \Big( \frac{e^{\Delta\tau A_\tau} - I}{\Delta\tau} - A_\tau \Big) e^{aB} U(r,t) x \Big\|_E : \tau, r \in [s,t), \Delta\tau < \max_j \Delta t_j \Big\}
\end{aligned}
\tag{3.12}
$$

The first term converges to zero by (3.11). We prove now the convergence of the second term to zero. By commuting of $e^{aB}$ with every $e^{tA_\xi}$, we get

$$
\begin{aligned}
&\Big\| \frac{U(\tau,\tau+\Delta\tau) - e^{\Delta\tau A_\tau}}{\Delta\tau} e^{aB} U(r,t)\, x \Big\|_E \\
&\qquad\qquad \leqslant \|e^{aB}\|_{E\to E} \Big\| \frac{U(\tau,\tau+\Delta\tau) - e^{\Delta\tau A_\tau}}{\Delta\tau} U(r,t) x \Big\|_E .
\end{aligned}
$$

Let $y \in F$. We use the well-known identity

$$
U(\tau,\tau+\Delta\tau) - e^{\Delta\tau A_\tau} = \int_0^{\Delta\tau} e^{sA_\tau} U(\tau+s,\tau+\Delta\tau)\big(A_{\tau+s} - A_\tau\big) U(\tau+s,\tau+\Delta\tau) ds
\tag{3.13}
$$

By stability of $A_t$, we have

$$
\frac{1}{\Delta\tau}\big(U(\tau,\tau+\Delta\tau) - e^{\Delta\tau A_\tau}\big) y \leqslant M\, e^{\Delta\tau\omega} \sup_{s\in[0,\Delta\tau]} \|(A_{\tau+s} - A_\tau)\mathcal{U}(\tau+s,\tau+\Delta\tau) y\|_E .
$$

By assumption 7, the convergence of the right-hand side holds uniformly in $\tau$. As $\Delta\tau \to 0$, the convergence (3.13) holds for every $y \in F$ and uniformly in $\tau$. Since the set $U(r,t)x$ is compact in $F$, by the same argument as before, the convergence

$$
\frac{1}{\Delta\tau}\big(U(\tau,\tau+\Delta\tau) - e^{\Delta\tau A_\tau}\big) U(r,t)\, x \to 0, \quad \text{as} \quad \Delta\tau \to 0,
$$

holds uniformly in $\tau$ and $r$. Thus, the convergence of the second term is proved. Consider the third term in (3.12). As before, we have

$$\Big\|\Big(\frac{e^{\Delta\tau A_\tau}-I}{\Delta\tau}-A_\tau\Big)\,e^{aB}U(r,t)\,x\Big\|_E \leqslant \|e^{aB}\|_{E\to E}\Big\|\Big(\frac{e^{\Delta\tau A_\tau}-I}{\Delta\tau}-A_\tau\Big)\,U(r,t)x\Big\|_E .$$

Again, by compactness of the set $\{U(t,r)\,x\}$ and Banach-Steinhaus theorem, it suffices to show that for each $x\in F$,

$$\Big(\frac{e^{\Delta\tau A_\tau}-I}{\Delta\tau}-A_\tau\Big)x \tag{3.14}$$

converges to zero uniformly in $\tau$, as $\Delta\tau\to 0$. Suppose first that $A_\tau$ are bounded operators, and $\|A_\tau\|\leqslant L$, where $L$, is independent of $\tau$. Then,

$$\Big\|\frac{e^{\Delta\tau A_\tau}-I}{\Delta\tau}-A_\tau\Big\|\leqslant\sum_{n=2}^{\infty}\frac{L^n(\Delta\tau)^{n-1}}{n!}\to 0, \quad \text{as} \quad \Delta\tau\to 0\ .$$

In the general case of the generators $A_t$, consider the bounded approximants $A_t^{(n)}$ defined above. Note that $\|e^{tA'_\xi}\|_{E\to E}\leqslant M$. From the proof of Lemma 21, it follows that $\|kR(k,A'_\xi)\|_{E\to E}<M$. This implies that $\sup_{\xi\in[s,t]}\|A_\xi^{(k)}\|_{E\to E}\leqslant \sup_{\xi\in[s,t]}\|A'^{(k)}_\xi\|_{E\to E}+\omega\leqslant k(M+1)+\omega\ (\stackrel{\text{den}}{=} M_k)$, and hence, for each fixed $k$, $\Big(\frac{e^{\Delta\tau A_\tau^{(k)}}-I}{\Delta\tau}-A_\tau^{(k)}\Big)\,x$ converges to zero uniformly in $\tau$. Fix an arbitrary $\varepsilon>0$. Find a number $k$ so that for all $\tau$ and $\Delta\tau$,

$$\|\big(A_\tau-A_\tau^{(k)}\big)\,x\|_E<\varepsilon \quad \text{and} \quad \Big\|\frac{1}{\Delta\tau}\big(e^{\Delta\tau A_\tau}-e^{\Delta\tau A_\tau^{(k)}}\big)\,x\Big\|_E<\varepsilon\ .$$

For the last inequality, we can find such $k$ by Lemma 23. Thus, we have that as $\Delta\tau\to 0$, the upper limit of (3.14) is smaller than $2\,\varepsilon$. Since $\varepsilon>0$ was chosen arbitrary, the limit of (3.14) exists and is equal to zero. Thus, the convergence is proved for the elements of the form $e^{aB}x$, where $x\in D$, $a>0$. But the set of such elements is dense in $E$. By the fact that $Q_{t_{i-1}t_i}$ are contractions and Banach-Steinhaus theorem, the convergence holds for all $x\in E$. The theorem is proved. □

LEMMA 24. *Let $A_1,\dots,A_n$ be a stable system of commuting generators satisfying (3.4), $\theta_1,\dots,\theta_n$ be non-negative numbers, $D\subset\cap_i D(A_i)$ be dense in $E$. Suppose that the operator $(\theta_1A_1+\dots+\theta_nA_n,D)$ is closed. Then, $e^{\theta_iA_i}$ and $e^{\theta_jA_j}$ commute for each $i$ and $j$, the operator $(\theta_1A_1+\dots+\theta_nA_n,D)$ generates a strongly continuous semigroup, and*

$$e^{t\sum_{i=1}^n\theta_iA_i}=\prod_{i=1}^n e^{t\theta_iA_i}. \tag{3.15}$$

*Proof.* By definition, the commutation of unbounded operators denotes commutation of their resolvents. By the Post-Widder Inversion Formula [13],

$$e^{tA} = \lim_{n\to\infty} \left(\tfrac{n}{t} R(\tfrac{n}{t}, A)\right)^n .$$

Hence, due to the commutation of the resolvents, the exponents of the commuting operators commute. Further, differentiating the product $\prod_{i=1}^n e^{t\theta_i A_i}x$ in $t$, we get that it is a solution of the Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t) \\ u(0) = x \end{cases}$$

with $A = \sum_{i=1}^n \theta_i A_i$. According to [13], in order to show that $\sum_{i=1}^n \theta_i A_i$ generates a strongly continuous semigroup, it suffices to show that for every sequence $D \ni x_n \to 0$, one obtains that $\prod_{i=1}^n e^{t\theta_i A_i} x_n \to 0$ uniformly in compact intervals $[0, t_0]$, which obviously holds by stability. Hence, $\sum_{i=1}^n \theta_i A_i$ generates a semigroup and the equality (3.15) holds. The lemma is proved. □

LEMMA 25. *Let $\{A_t\}$ be a stable system of generators, and let the operators $A_t$ commute for different $t$. Further, assume that for all $s, t \in [S, T], s \leqslant t$, the operator $\int_s^t A_\xi d\xi$ [4] generates a semigroup. Then, the evolution family $U(s,t)$ solving the Cauchy problem* (3.5)*, has the form*

$$U(s,t) = e^{\int_s^t A_\xi d\xi}$$

*Proof.* Let $\mathcal{P} = \{s = \tau_0 < \cdots < \tau_n = t\}$, $\xi_j \in [\tau_{j-1}, \tau_j)$. According to [14], the evolution family $U(s,t)$ solving the Cauchy problem (3.5), is the pointwise limit of

$$U^{\mathcal{P}}(s,t) = exp\{(\tau_n - \tau_{n-1})A_{\xi_n}\} \ldots exp\{(\tau_1 - \tau_0)A_{\xi_1}\} ,$$

which is equal to

$$exp\{(\tau_1 - \tau_0)A_{\xi_1} + \cdots + (\tau_n - \tau_{n-1})A_{\xi_n}\}$$

by Lemma 24.

Now let us assume that $A$ and $B$ generate strongly continuous semigroups and the semigroup generated by $B$, satisfy (3.4) with the constants $M$ and $(t-s)\omega$. We have

$$e^A x - e^B x = \int_0^1 \frac{d}{d\xi}\left(e^{(1-\xi)A} e^{\xi B} x\right) d\xi = \int_0^1 e^{(1-\xi)A} e^{\xi B}(Ax - Bx) d\xi.$$

Let $M_1$ be such that $\|e^{(1-\xi)A}\| < M_1$ for all $\xi \in [0,1]$. Taking into account that $\|e^{\xi B}\| \leqslant Me^{\xi(t-s)\omega} \leqslant Me^{(t-s)\omega}$, we get

$$\|e^A x - e^B x\|_E \leqslant M_1 M e^{(t-s)\omega} \|Ax - Bx\|_E. \tag{3.16}$$

[4] The operator $\int_s^t A_\xi d\xi$ is defined on $F$ as follows: $(\int_s^t A_\xi d\xi)x = \int_s^t (A_\xi x) d\xi$.

Apply the last inequality for $A = \int_s^t A_\xi d\xi$ and $B = \sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}$. Since the system $A_\xi$ is stable,, the norm of $e^{\xi \sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}}$ can be estimated as

$$\|e^{\xi \sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}}\| \leqslant M e^{\xi (t-s)\omega} , \tag{3.17}$$

which also implies the uniform boundedness of $e^{\sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}}$. Further, let $x \in F$. The sum $\sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i} x$ converges to $\int_s^t A_\xi d\xi \, x$ by definition of the integral $\int_s^t A_\xi d\xi$. This together with the inequality (3.16) implies that as the mesh of $\mathcal{P}$ tends to 0, the sequence $e^{\sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}} x$ converges to $e^{\int_s^t A_\xi d\xi} x$. By uniform boundedness of $e^{\sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}}$ and Banach-Steinhaus theorem, the convergence holds for all $x \in E$. On the other hand $e^{\sum_{i=1}^n (\tau_i - \tau_{i-1}) A_{\xi_i}} x$ converges to the solution $U(s,t)\, x$ of the Cauchy problem (3.5) as it was mentioned above. The lemma is proved. □

THEOREM 16. *Let $A_t$ be generators of strongly continuous semigroups, $B$ another generator of a strongly continuous semigroup. We assume that the following assumptions are fulfilled:*

1) *$D = F \bigcap D(B)$ is dense in $E$;*
2) *the functions $[S,T] \to E$, $t \mapsto A_t x$, are continuous for all $x \in F$;*
3) *for different $t$, the operators $A_t$ commute; $B$ commutes with every $A_t$;*
4) *for all $s, t \in [S,T], s \leqslant t$, the operators $\int_s^t A_\xi d\xi$ generate strongly continuous semigroups;*
5) *there exists a sequence of closed partial sums approximating $\int_s^t A_\xi d\xi$;*
6) *$\{A_t\}$ is a stable system of generators with the stability constants $M$ and $\omega$;*
7) *the Cauchy problem (3.5) is well-posed on $F$.*

*Let $Q_{t_1,t_2}$, $t_1, t_2 > 0$, be a two-parameter family of contractions on $E$ such that*

$$\frac{Q_{\tau,\tau+\Delta\tau} - I}{\Delta\tau} e^{aB} x \to A_\tau \, e^{aB} x$$

*as $\Delta\tau \to 0$, for all $x \in D$, for all $a > 0$, and uniformly in $\tau$. Let $S \leqslant s < t \leqslant T$, $\{s = t_0 < t_1 < \ldots < t_n = t\}$ be a partition of the interval $[s,t]$ so that $\max \Delta t_j \to 0$ as $n \to \infty$, where $\Delta t_j = t_{j+1} - t_j$ . Then, for all $x \in E$,*

$$Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n} x \to e^{\int_s^t A_\xi d\xi} \, x$$

*as $n \to \infty$.*

*Proof.* The statement of the theorem is a direct consequence of Theorem 15 and Lemma 25. Indeed, as a set $D$ in the condition 6 of Theorem 15, we can take the set $G$ itself. The condition 7 holds by commuting of the operators $A_t$ and $B$. □

Further, we will apply this result to generators of contraction semigroups satisfying the condition $D(A_t) = Y$ for all $t$ and to the functions $[S, T] \to E, t \mapsto A_t x$ from $C^1(\mathbb{R}, E)$ for all $x \in Y$. This result is due to Kato [17].

## 3.3 Asymptotic in $t$ for the integral of the form $(2\pi t)^{-\frac{d}{2}} \int_M g(y) e^{-\frac{|x-y|^2}{2t}} dy$.

PROPOSITION 3. *Let $\imath$ be an isometric imbedding of the manifold $M$ into $\mathbb{R}^m$, $g \in C^2(M)$. Then,*

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz) = g(y) + \frac{t}{8} g(y)\big(c(y) - scal(y)\big) - \frac{t}{2} \Delta_M g(y) + tR(t, y) \ , \tag{3.18}$$

*where $|R(t, y)| < Kt^{1/2}$ and $K$ is a constant which is independent of $y$; the function $c(y)$ has the form:*

$$c(y) = \sum_{i,j} \sum_\alpha \Big( \frac{\partial^2 i^\alpha}{\partial x^i \partial x^j} \Big)^2 (0) \ , \tag{3.19}$$

*where $x^i$ are normal coordinates in some neighborhood $U_y$ of the point $y$ provided by a homeomorphism of $U_y$ on some neighborhood of zero $U$ in $\mathbb{R}^d$, $i$ is the embedding $\imath$ written in the local coordinates $x^i$. Independently of local coordinates, $c(y)$ can be written as*

$$c(y) = -\frac{1}{2} \Delta_M \Delta_M \left. |y - \cdot|^2 \right|_y - \frac{1}{3} scal(y) \ , \tag{3.20}$$

*and hence, it depends only on the imbedding $\imath$.*

*Proof.* It is well known that $|\imath(z) - \imath(y)|^2 = d(y, z)^2 + \varphi(y, z)$, where $d$ is a geodesic distance in $M$ and $\varphi(y, z) = O(d(y, z)^4)$. Let $\psi_y : U \to U_y$ be the diffeomorphism providing the normal coordinates in $U_y$, $f_y(x) = \varphi(y, \psi_y(x))$, $h_y(x) = \sqrt{\det g_{ij}(x)}\, g(\psi_y(x))$ where $g_{ij}$ is the metric tensor. We have

$$\int_{U_y} e^{-\frac{|z-y|^2}{2t}} g(z) \lambda_M(dz) = \int_{U_y} e^{-\frac{d(y,z)^2 + \varphi(y,z)}{2t}} g(z) \lambda_M(dz) = \int_U e^{-\frac{|x|^2 + f_y(x)}{2t}} h_y(x) dx \ .$$

By results of [25],

$$\frac{1}{(2\pi t)^{\frac{d}{2}}}\int_U e^{-\frac{|x|^2+f_y(x)}{2t}} h_y(x)dx = h_y(0)+\frac{t}{2}\Delta h_y(0)-\frac{t}{16}h_y(0)\Delta\Delta f_y(0)+\bar{R}_t \quad (3.21)$$

where $|\bar{R}_t| < \bar{K}t^{1/2}$ and $\bar{K}$ is a constant. As it was mentioned in [25], the sets $\{\varphi(y,\cdot) : y \in M\}$ and $\{g_{ij} : y \in M\}$ are uniformly in $y$ bounded in $C^5(M)$ and $C^3(U)$ respectively, and the neighborhood $U$ can be chosen independently of $y$. This implies that the constant $\bar{K}$ can be also chosen independently of $y$. Note that $h_y(0) = g(y)$, and as it was calculated in [25], $\Delta h_y(0) = -\Delta_M u(y) - \frac{1}{3}u(y)scal(y)$. Calculate $\Delta\Delta f_y(0)$. Note that $\Delta\Delta\, d(y,\psi_y(x))^2 = \Delta\Delta|x|^2 = 0$, and hence $\Delta\Delta f_y(0) = \Delta\Delta\big(|\imath\circ\psi_y(x) - \imath(y)|^2\big)|_{x=0}$. We get

$$\Delta\Delta f_y(0) = \Delta\Delta\sum_\alpha (i^\alpha(x) - i^\alpha(0))^2|_{x=0}\ .$$

It is easy to see that the result of the calculation of the right-hand side does not depend on $i(0)$, so we may assume that $i(0) = 0$. It is well known (see for example [16]) that

$$g_{ij}(x) = \sum_\alpha \frac{\partial i^\alpha}{\partial x^i}\frac{\partial i^\alpha}{\partial x^j}(x)\ .$$

From this we get

$$\sum_{i,j}\frac{\partial^2 g_{ij}}{\partial x^i\partial x^j} = \sum_{i,j}\sum_\alpha\Big(\frac{\partial^2 i^\alpha}{\partial x^i\partial x^j}\Big)^2 + 2\sum_{i,j}\sum_\alpha\frac{\partial i^\alpha}{\partial x^j}\frac{\partial^3 i^\alpha}{(\partial x^i)^2\partial x^j} + \sum_{i,j}\sum_\alpha\frac{\partial^2 i^\alpha}{(\partial x^i)^2}\frac{\partial^2 i^\alpha}{(\partial x^j)^2}\ . \quad (3.22)$$

Further

$$\sum_\alpha\frac{\partial^2(i^\alpha(x))^2}{(\partial x^j)^2} = 2\sum_\alpha\Big(\frac{\partial i^\alpha(x)}{\partial x^j}\Big)^2 + i^\alpha(x)\frac{\partial^2 i^\alpha(x)}{(\partial x^j)^2} = 2\,g_{jj} + 2\sum_\alpha i^\alpha(x)\frac{\partial^2 i^\alpha(x)}{(\partial x^j)^2}\ .$$

Differentiating the last equality twice in $x_i$, taking the sum in $i$ and $j$, and using (3.22), we get

$$\begin{aligned}\Delta\Delta\sum_\alpha(i^\alpha(x))^2|_{x=0} &= 2\Big(\sum_{i,j}\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) + 2\sum_{i,j}\sum_\alpha\frac{\partial i^\alpha}{\partial x^j}\frac{\partial^3 i^\alpha}{(\partial x^i)^2\partial x^j}(0)\\ &\quad + \sum_{i,j}\sum_\alpha\frac{\partial^2 i^\alpha}{(\partial x^i)^2}\frac{\partial^2 i^\alpha}{(\partial x^j)^2}(0)\Big)\\ &= 2\Big(\sum_{i,j}\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) + \sum_{i,j}\frac{\partial^2 g_{ij}}{\partial x^i\partial x^j}(0) - \sum_{i,j}\sum_\alpha\Big(\frac{\partial^2 i^\alpha}{\partial x^i\partial x^j}\Big)^2(0)\Big).\end{aligned}$$

In normal coordinates, the metric tensor has the form [20]

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj} x^k x^l + O(|x|^4)$$

where $R$ is the curvature tensor. From this, we have

$$\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) = \frac{2}{3} R_{jiij}(0) = -\frac{2}{3} R_{ijij}(0) \ , \quad \frac{\partial^2 g_{ij}}{\partial x^i \partial x^j}(0) = \frac{1}{3}(R_{iijj}(0) + R_{ijij}(0)) \ .$$

In normal coordinates, $R_{ijij}(x) = scal(x)$ (see [16]). The scalar curvature does not depend on choice of local coordinates, and hence, it only depends on point on the manifold. Taking into account this, we get

$$\sum_{i,j} \frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) + \sum_{i,j} \frac{\partial^2 g_{ij}}{\partial x^i \partial x^j}(0) = -\frac{1}{3} scal(y) \ .$$

Finally,

$$\Delta\Delta f_y(0) = \Delta\Delta \sum_{\alpha} (i^\alpha(x))^2|_{x=0} = -2(\tfrac{1}{3} scal(y) + c(y)) \ ,$$

where $c(y)$ is defined by (3.19). This implies also (3.20) if we take into account that the operator $\Delta_M$ in normal coordinates becomes the opposite to the ordinary Laplace operator. Choose now neighborhoods $U_y$ of the form: $U_y = \{z \in M : |z - y| < \varepsilon\}$, where $\varepsilon$ is chosen as follows. As it was mentioned in [25], the diameter of the neighborhood of the point $y$ where we can introduce the normal coordinate system, bounded from zero, say by $\varepsilon$. Define each $U_y$ with the help of this $\varepsilon$. We have

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{M \setminus U_y} g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz) \leqslant \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{\varepsilon^2}{2t}} \int_M |g(z)| \lambda_M(dz) < t^{3/2} \quad (3.23)$$

for $t$ smaller than some number $t_0$. The last estimate holds uniformly in $y$ for all $t < t_0$. Substituting the expressions for $\Delta h_y(0)$ and $\Delta\Delta f_y(0)$ into (3.21) and taking into account (3.23), we get (3.18) with $R(t, y)$ satisfying $|R(t, y)| < Kt^{1/2}$, where the constant $K$ is independent of $y$. □

COROLLARY 3. *Let $g \in C^2(M)$. Then, we have the following asymptotic:*

$$\frac{\int_M g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz)} = g(y) - \frac{t}{2} \Delta_M g(y) + t R_1(t, y) \ , \quad (3.24)$$

*where $|R_1(t, y)| < K_1 t^{1/2}$, and $K_1$ is a constant which is independent of $y$.*

*Proof.* The statement of the corollary easily follows from the the above proposition applied to the functions $g(y)$ and $g(y) \equiv 1$, respectively. $\square$

PROPOSITION 4. *Let $\imath$ be an isometrical embedding of the manifold $M$ into $\mathbb{R}^m$, $g \in C^2(M)$, $y \in M$, $0 < t < t_1 < 1$, $u_1$ and $u_2$ be such that $|u_2 - u_1| < t_1^\alpha$, $\alpha > 0$. Further, let $\mathrm{Pr}_M$ be the mapping of projection on the manifold $M$ along the normal subspaces of the manifold, which is defined in the some neighborhood of the manifold, $(u_2 - u_1)_M^y$ and $(u_2 - u_1)_\perp^y$ be defined as*

$$(u_2-u_1)_M^y = \mathrm{Pr}_M(y+u_2-u_1)-y, \quad (u_2-u_1)_\perp^y = y+u_2-u_1-\mathrm{Pr}_M(y+u_2-u_1).$$

*Then, the asymptotic*

$$\frac{\int_M g(z)e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}}\lambda_M(dz)}{\int_M e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}}\lambda_M(dz)} = g(y+(u_2-u_1)_M^y) - \frac{t}{2}\Delta_M g(y+(u_2-u_1)_M^y)$$
$$+ g(y+(u_2-u_1)_M^y)\mathcal{R}(y, u_2-u_1) + tR_2(t, t_1, y, u_2-u_1), \quad (3.25)$$

*holds, and for the rest terms, the following estimates $|R_2(t, t_1, y, u_2-u_1)| < K_2 t_1^\alpha$ ($K_2$ is a constant) and*

$$\mathcal{R}(y, u_2-u_1) = -\frac{1}{4}(\Delta_M \imath(y+(u_2-u_1)_M^y), (u_2-u_1)_\perp^y)^2$$
$$+\sum_{n=3}^{N}\sum k(n)\prod_{l_1+\cdots+l_s=n} D_M^{2,l_i}(\imath(\cdot), (u_2-u_1)_\perp^y)^{l_i}\big(y+(u_2-u_1)_M^y\big),$$

*hold, where $D_M^{2,l_i}$ are differential operators on the manifold which have the form $D_M^{2,l_i} = (\wedge_{k=1}^{l_i}\nabla_M^{(i_k)}, \wedge_{k=1}^{l_i}\nabla_M^{(j_k)})$; the operators are applied to the product of $l_i$ function, $i_k$ and $j_k$ are numbers from 1 to $l_i$, which have the meaning of the number of function in the product on which the corresponding operator acts; the index 2 says that $i_k$ and $j_k$ take the same values exactly 2 times, $k(n)$ are rational functions; the second sum in the last term contains a finite number of items, the number $N$ is chosen so that $t_1^{(N-1)\alpha} < t$.*

*Proof.* We have

$$\frac{\int_M g(z)\,e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}}\lambda_M(dz)}{\int_M e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}}\lambda_M(dz)} = \frac{\int_M g(z)\,e^{-\frac{|z-(y+(u_2-u_1)_M^y)-(u_2-u_1)_\perp^y|^2}{2t}}\lambda_M(dz)}{\int_M e^{-\frac{|z-(y+(u_2-u_1)_M^y)-(u_2-u_1)_\perp^y|^2}{2t}}\lambda_M(dz)}$$
$$= \frac{\int_M g(z)\,e^{-\frac{|z-(y+(u_2-u_1)_M^y)|^2}{2t}}\,e^{\frac{(z-(y+(u_2-u_1)_M^y),(u_2-u_1)_\perp^y)}{t}}\lambda_M(dz)}{\int_M e^{-\frac{|z-(y+(u_2-u_1)_M^y)|^2}{2t}}\,e^{\frac{(z-(y+(u_2-u_1)_M^y),(u_2-u_1)_\perp^y)}{t}}\lambda_M(dz)}$$

Everywhere below, $K$, $R$, $\mathcal{K}$ are constants, not necessarily the same.

We choose an $\varepsilon$-neighborhood $U_{y+(u_2-u_1)^y_M}$ of the point $y+(u_2-u_1)^y_M$, and fix the system of normal coordinates. We have

$$\frac{1}{(2\pi t)^{\frac{d}{2}}}\int\limits_{M\setminus U_{y+(u_2-u_1)^y_M}} g(z)\, e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}\, e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz) < \frac{K}{(2\pi t)^{\frac{d}{2}}}\, e^{\left(\frac{R}{t\,t_1^{-\alpha}}-\frac{\varepsilon^2}{2t}\right)} .$$

Let $h_y$ and $f_y$, denote the same as in the proof of Proposition 3, but relative to the point $y+(u_2-u_1)^y_M$, more precisely, $f_y(x)=\varphi(y,\psi_y(x))$, $h_y(x)=\sqrt{\det g_{ij}(x)}\, g(\psi_y(x))$, where $\psi_y : U\to U_{y+(u_2-u_1)^y_M}$ is a homeomorphism providing normal coordinates $x_k$ in the neighborhood $U_{y+(u_2-u_1)^y_M}$, $g_{ij}$ is the metric tensor. Further, we calculate integrals in the numerator and denominator relative to the system of normal coordinates $x_k$ in the neighborhood $U_{y+(u_2-u_1)^y_M}$ of the point $y+(u_2-u_1)^y_M$. Note that $z-(y+(u_2-u_1)^y_M)=x+\tilde f(x)$, where $x$ belongs to $T_{y+(u_2-u_1)^y_M}$, the tangent space to the manifold $M$ at the point $y+(u_2-u_1)^y_M$, $\tilde f(x)$ is orthogonal to $T_{y+(u_2-u_1)^y_M}$ and such that $\tilde f(0)=0$, $\frac{\partial \tilde f}{\partial x_k}(0)=0$ for all $k$. Since $(u_2-u_1)^y_\perp$ is orthogonal to $T_{y+(u_2-u_1)^y_M}$, we have

$$\begin{aligned}(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp) &= (\tilde f(x),(u_2-u_1)^y_\perp)\\ &=\frac{1}{2}\Big(\frac{\partial^2\tilde f}{\partial x^i\partial x^j},(u_2-u_1)^y_\perp\Big)x^ix^j+\frac{1}{6}\Big(\frac{\partial^3\tilde f}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_3}},(u_2-u_1)^y_\perp\Big)x^{i_1}x^{i_2}x^{i_3}\\ &\qquad +\|\tilde f\|_4\,\theta(x)\,|x|^4\,t_1^{\alpha}\ .\end{aligned}$$

Let $\nu_t$ denote the Gauss distribution with the covariance matrix $tI$. Let $t$ be sufficiently small, so that for all $0\leqslant k\leqslant 3N$,

$$\int_{\mathbb{R}^d\setminus U}|x|^k\nu_t(dx)<t^N\ . \tag{3.26}$$

Further, let $\theta_i$ denote functions satisfying the condition $|\theta_i| \leqslant 1$. We have,

$$
\begin{aligned}
e^{\frac{(\tilde f(x),(u_2-u_1)^y_\perp)}{t}} &= 1 + \frac{1}{2}\Big(\frac{\partial^2 \tilde f}{\partial x^i \partial x^j}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^i x^j}{t} \\
&+ \frac{1}{6}\Big(\frac{\partial^3 \tilde f}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_3}}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}x^{i_2}x^{i_3}}{t} \\
&+ \sum_{n=2}^{N-1}\frac{1}{2^n n!}\prod_{s=1}^{n}\Big(\frac{\partial^2 \tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n}}}{t^n} \\
&+ \sum_{n=2}^{N-1}\frac{1}{3{\cdot}2^n n!}\prod_{s=1}^{n-1}\Big(\frac{\partial^2 \tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}\,,(u_2-u_1)^y_\perp\Big) \\
&\qquad \times\Big(\frac{\partial^3 \tilde f}{\partial x^{i_{2n-1}}\partial x^{i_{2n}}\partial x^{i_{2n+1}}}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n+1}}}{t^n} \\
&+ \|\tilde f\|_4\,\theta(x)\,\frac{|x|^4}{t}\,t_1^\alpha + \|\tilde f\|_3^N\, t{\cdot}t_1^\alpha \textstyle\sum_{s=0}^{N} k(s,N)\theta_s(x)\dfrac{|x|^{2N+s}}{t^N}\;.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(2\pi t)^{\frac{m}{2}}}\int_{\mathbb{R}^d}\exp\Big(-\frac{|x|^2+f_y(x)}{2t}\Big)h_y(x)\,e^{\frac{(\tilde f(x),(u_2-u_1)^y_\perp)}{t}}dx \\
&= \int_{\mathbb{R}^d}\Big(\Big(1-\frac{1}{48t}\,\frac{\partial^4 f_y}{\partial x^{i_1}\cdots\partial x^{i_4}}(0)\,x^{i_1}\cdots x^{i_4} + \frac{\|f_y\|_5\theta_1(x)|x|^5}{t}\Big) \\
&\times\Big(h_y(0)+\frac{\partial h_y}{\partial x^{i_1}}(0)x^{i_1}+\frac{1}{2}\,\frac{\partial^2 h_y}{\partial x^{i_1}\partial x^{i_2}}(0)\,x^{i_1}x^{i_2}+\|h_y\|_3\,\theta_2(x)\,|x|^3\Big) \\
&\times\Big(1+\sum_{n=1}^{N-1}\frac{1}{2^n n!}\prod_{s=1}^{n}\Big(\frac{\partial^2 \tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}(0)\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n}}}{t^n} \\
&\quad+\sum_{n=1}^{N-1}\frac{1}{3{\cdot}2^n n!}\prod_{s=1}^{n-1}\Big(\frac{\partial^2 \tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}(0)\,,(u_2-u_1)^y_\perp\Big) \\
&\quad\times\Big(\frac{\partial^3 \tilde f}{\partial x^{i_{2n-1}}\partial x^{i_{2n}}\partial x^{i_{2n+1}}}(0)\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n+1}}}{t^n} \\
&+\|\tilde f\|_4\,\theta(x)\,\frac{|x|^4}{t}\,t_1^\alpha+\|\tilde f\|_3^N\,t{\cdot}t_1^\alpha \textstyle\sum_{s=0}^{N}k(s,N)\theta_s(x)\dfrac{|x|^{2N+s}}{t^N}\Big)\nu_t(dx)\;. \qquad (3.27)
\end{aligned}
$$

The number $N$ is fixed so that $t_1^{\alpha N} < t\,t_1^\alpha$. The norms $\|f\|_k$ are introduced in the same way as in [25]. By condition (3.26), replacing the integration over the space $\mathbb{R}^d$ with the integration over the neighborhood $U$, gives the rest term which is not bigger than $K\,t\,t_1^{\alpha}$. Performing the integration in (3.27), we get

$$\frac{1}{(2\pi t)^{\frac{m}{2}}}\int_{\mathbb{R}^d}\exp\Bigl(-\frac{|x|^2+f_y(x)}{2t}\Bigr)h_y(x)\,e^{\frac{(\tilde f(x),(u_2-u_1)^y_\perp)}{t}}dx$$
$$=h_y(0)+\frac{t}{2}\Delta h_y(0)-\frac{t}{16}h_y(0)\Delta\Delta f_y(0)+\frac{1}{2}(\Delta\tilde f(0),(u_2-u_1)^y_\perp)$$
$$+\sum_{n=2}^{N-1}\frac{1}{2^n n!}\sum\prod_{l_1+\dots+l_s=n}D^{2,l_i}(\tilde f(\cdot),(u_2-u_1)^y_\perp)^{l_i}(0)+\tilde R(t,t_1),\quad(3.28)$$

Here $D^{2,l_i}$ are differential operators of the form $D^{2,l_i}=(\wedge_{k=1}^{l_i}\nabla^{(i_k)},\wedge_{k=1}^{l_i}\nabla^{(j_k)})$; the operators are applied to the product of $l_i$ functions, $i_k$ and $j_k$ are the numbers from 1 to $l_i$, which have the meaning of the number of the function in the product, on which the corresponding operator acts, the index 2 says that $i_k$ and $j_k$ take the same values exactly 2 times, the second sum contains a finite number of terms, $\tilde R(t,t_1)$ is the rest term, which we will estimate later. Note that $\Delta\tilde f(0)=\Delta_M\imath(y+(u_2-u_1)^y_M)$, $D^{2,l_i}(\tilde f(\cdot),(u_2-u_1)^y_\perp)^{l_i}(0)=D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}(y+(u_2-u_1)^y_M)$. Further, applying Proposition 3, we obtain

$$\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}g(z)\,e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz)=g(y+(u_2-u_1)^y_M)$$
$$-\frac{t}{2}\Delta_M g(y+(u_2-u_1)^y_M)+\frac{t}{8}g(y+(u_2-u_1)^y_M)$$
$$\times\bigl(c(y+(u_2-u_1)^y_M)-scal(y+(u_2-u_1)^y_M)\bigr)-\frac{1}{2}(\Delta_M\imath(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)$$
$$+\sum_{n=2}^{N-1}\frac{1}{2^n n!}\sum\prod_{l_1+\dots+l_s=n}D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}(y+(u_2-u_1)^y_M)+\tilde R(t,t_1)\ .$$

Finally, we have

$$\frac{\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}g(z)\,e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz)}{\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}\,e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz)}$$
$$=g(y+(u_2-u_1)^y_M)\bigl(1-\frac{1}{4}(\Delta_M\imath(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)^2$$
$$+\sum_{n=3}^{N-1}\sum k(n)\prod_{l_1+\dots+l_s=n}D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}(y+(u_2-u_1)^y_M)\bigr)$$
$$-\frac{t}{2}\Delta_M g(y+(u_2-u_1)^y_M)+R(t,t_1),$$

$k(n)$ are rational functions, $R(t,t_1)$ is the rest term, which, by inequality $\int_{\mathbb{R}^d}|x|^{2N+s}\nu_t(dx)=\mathcal{K}\,t^{N+\frac{s}{2}}$, is not bigger than $K\,t\,t_1^{\alpha}$. □

Corollary 4. *Let $g \in C^2(M)$, $y \in M$, $\psi$ is a differentiable function with bounded derivatives, and such that $\psi(0) = 0$. Define $\psi_M(t, y) = \mathrm{Pr}_M(y + \psi(t))$. The following asymptotic holds:*

$$\frac{\int_M g(z) e^{-\frac{|z-y-\psi(t)|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-\psi(t)|^2}{2t}} \lambda_M(dz)} = g(y + \psi_M(t)) - \frac{t}{2}\Delta_M g(y) + tR_2(t, y) \ , \tag{3.29}$$

*where $|R_2(t, y)| < K_2 t^{1/2}$ for differentiable functions $\psi$, $K_2$ is a constant which is independent of $y$.*

## 3.4 Application of Chernoff's theorem to construction of non-homogeneous processes on manifolds

Proposition 5. *Let $w^\alpha_{s,\varphi}(t) = b^\alpha t + w^\alpha(st)$, where $w^\alpha(t)$ is an $m$-dimensional canonical wiener process, $b \in \mathbb{R}^m$, $\tilde{L}_\alpha$ canonical horizontal vector fields [15]. Then, the solution of the system of stochastic differential equations*

$$\begin{cases} dr(t) = \tilde{L}_\alpha(r(t)) \circ dw^\alpha_{s,\varphi}, \\ r(0) = r \ , \end{cases} \tag{3.30}$$

*has the form $r(t) = (X^i(t), e^i_\alpha(t))$, where the process $X^i(t)$ has the generator*

$$Af = (\nabla_M f, b)_{\mathbb{R}^m} - \frac{s}{2}\Delta_M f \ .$$

*Proof.* Let $r(t) = (X^i(t), e^i_\alpha(t))$ be the solution of the system (3.30). Find the generator of $X(t)$. Consider the function $f(r) = f(x)$ for $r = (x, e)$. We have

$$\begin{aligned} f(X(t)) - f(X(0)) &= f(r(t)) - f(r(0)) \\ &= \int_0^t (\tilde{L}_\alpha f)(r(\xi)) \circ dw^\alpha_{s,\varphi}(\xi) \\ &= \int_0^t \tilde{L}_\alpha f(r(\xi)) dw^\alpha_s(\xi) + \int_0^t \tilde{L}_\alpha f(r(\xi)) b^\alpha d\xi \\ &\quad + \frac{s}{2}\int_0^t \sum_{\alpha=1}^d \tilde{L}_\alpha(\tilde{L}_\alpha f)(r(\xi)) d\xi \ , \end{aligned}$$

where $w_s(t) = w(st)$ is a multiple Brownian motion in $\mathbb{R}^m$ with parameter $s$. From the definition of the generator, we have

$$Af = \sum_{\alpha=1}^{d} \tilde{L}_\alpha f \cdot b^\alpha + \frac{s}{2}\sum_{\alpha=1}^{d} \tilde{L}_\alpha(\tilde{L}_\alpha f).$$

An analogous expression was calculated in [15]. Using it, we get

$$Af = (\nabla_M f, b)_{\mathbb{R}^m} - \frac{s}{2}\Delta_M f \ .$$

□

Let $\mathcal{P}_1 = \{t_0 = 0 < t_1 < \ldots < t_n = 1\}$ be a partition of the interval $[0,1]$, $\varphi : [0,1] \to M - x$ a differentiable function such that $\varphi(0) = 0$. If $E$ is a LCS, then every $\omega \in E^1$ can be identified with the $n$-tuple $(\omega_1, \omega_2, \ldots, \omega_n) \in E^{t_1} \times E^{t_2-t_1} \times \ldots \times E^{t_n - t_{n-1}}$, where $\omega_j$ is defined on the interval $[0, t_j - t_{j-1}]$ by $\omega_j(t) = \omega(t_{j-1} + t)$. Define the function $\varphi_{t_{i-1}t_i}$ on the interval $[0, t_i - t_{i-1}]$ by

$$\varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1} + t) - \varphi(t_{i-1}) \ .$$

Define the measure $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ by

$$\int\limits_{C([0,1],\mathbb{R}^m)} h(\omega)\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,t_1],\mathbb{R}^m)} \mathbb{W}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int\limits_{C([0,t_2-t_1],\mathbb{R}^m)} \mathbb{W}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2)$$
$$\cdots \int\limits_{C([0,t_n-t_{n-1}],\mathbb{R}^m)} \mathbb{W}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) h(\omega_1, \omega_2, \ldots, \omega_n). \tag{3.31}$$

We want to apply Theorem 15 to the family of generators

$$A_t f = (\varphi'(t), \nabla_M f)_{\mathbb{R}^m} - \frac{s}{2}\Delta_M \ . \tag{3.32}$$

By Proposition 5, the operators (3.32) generate contraction semigroups.

THEOREM 17. *Let $\varphi$ be a differentiable function. Then, as the mesh of $\mathcal{P}_1$ tends to zero, the sequence of measures $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ converges weakly relative to the family of bounded continuous cylinder functions.*

*Proof.* Consider the integral

$$\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\, \mathbb{W}^z_{M,\psi,s,t}(d\omega) \ , \tag{3.33}$$

where $z \in M$, $\psi(0) = 0$, and the function $g$ is such that there exists a function $\tilde{g} : \mathbb{R}^m \to \mathbb{R}$, and $g(\omega) = \tilde{g}(\omega(t))$. From (3.1), it follows that

$$\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\, \mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\, \mathbb{I}_{\{\omega:\, \omega(t)\in U_\varepsilon(M)\}}(\omega)\, \mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega : \omega(t) \in U_\varepsilon(M)\}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega+\psi)\, \mathbb{I}_{\{\omega:\, \omega(t)\in U_\varepsilon(M)\}}(\omega+\psi)\, \mathbb{W}^z_{0,s}(d\omega)}{\mathbb{W}^z_{0,s}\{\omega : \omega(t) \in U_\varepsilon(M) - \psi(t)\}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\, \mathbb{I}_{\{\omega:\, \omega(t)\in U_\varepsilon(M)\}}(\omega) \mathbb{W}^{z+\psi(t)}_{0,s}(d\omega)}{\mathbb{W}^{z+\psi(t)}_{0,s}\{\omega : \omega(t) \in U_\varepsilon(M)\}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int\limits_{U_\varepsilon(M)} \tilde{g}(x_1) \mathbb{W}^{z+\psi(t)}_{0,s} \circ \pi_t^{-1}(dx_1)}{\mathbb{W}^{z+\psi(t)}_{0,s} \circ \pi_t^{-1}\big(U_\varepsilon(M)\big)}$$

$$= \frac{\int\limits_M e^{-\frac{|x_1-z-\psi(t)|^2}{2ts}}\, \tilde{g}(x_1)\, \lambda_M(dx_1)}{\int\limits_M e^{-\frac{|x_1-z-\psi(t)|^2}{2ts}}\, \lambda_M(dx_1)}\ .$$

Noting that $\varphi_{t_{i-1}t_i}(t_i - t_{i-1}) = \varphi(t_i) - \varphi(t_{i-1}) = \Delta\varphi(t_i)$, we can rewrite the integral (3.31) in the form

$$\frac{\int\limits_M e^{-\frac{|x_1-x-\Delta\varphi(t_1)|^2}{2st_1}} dx_1 \ldots \dfrac{\int\limits_M e^{-\frac{|x_{n-1}-x_{n-2}-\Delta\varphi(t_{n-1})|^2}{2s(t_{n-1}-t_{n-2})}} dx_{n-1} \dfrac{\int\limits_M e^{-\frac{|x_n-x_{n-1}-\Delta\varphi(t_n)|^2}{2s(t_n-t_{n-1})}} \tilde{h}(\bar{x}_{i_1,\ldots,i_k}) dx_n}{\int\limits_M e^{-\frac{|x_n-x_{n-1}-\Delta\varphi(t_n)|^2}{2s(t_n-t_{n-1})}} dx_n}}{\int\limits_M e^{-\frac{|x_{n-1}-x_{n-2}-\Delta\varphi(t_{n-1})|^2}{2s(t_{n-2}-t_{n-1})}} dx_{n-1}}}{\int\limits_M e^{-\frac{|x_1-x-\Delta\varphi(t_1)|^2}{2st_1}} dx_1}$$

where $h(w) = \tilde{h}(w(\tau_1), \ldots, w(\tau_k))$. It suffices to prove the convergence of this fraction for functions $h$ depending on $t$ only at one point, i.e. for which there exists a function $\tilde{h} : \mathbb{R}^m \to \mathbb{R}$ such that $h(w) = \tilde{h}(w(t))$. First we will consider only those partitions which contain the point $t$. Without loss of generality we can consider $t_n = t$. The above fraction can be written as follows:

$$\frac{\int\limits_M e^{-\frac{|x_1-x-\Delta\varphi(t_1)|^2}{2st_1}} dx_1 \ldots \frac{\int\limits_M e^{-\frac{|x_{n-1}-x_{n-2}-\Delta\varphi(t_{n-1})|^2}{2s(t_{n-1}-t_{n-2})}} dx_{n-1} \frac{\int\limits_M e^{-\frac{|x_n-x_{n-1}-\Delta\varphi(t_n)|^2}{2s(t_n-t_{n-1})}} \tilde{h}(x_n) dx_n}{\int\limits_M e^{-\frac{|x_n-x_{n-1}-\Delta\varphi(t_n)|^2}{2s(t_n-t_{n-1})}} dx_n}}{\int\limits_M e^{-\frac{|x_{n-1}-x_{n-2}-\Delta\varphi(t_{n-1})|^2}{2s(t_{n-2}-t_{n-1})}} dx_{n-1}}}{\int\limits_M e^{-\frac{|x_1-x-\Delta\varphi(t_1)|^2}{2st_1}} dx_1} \tag{3.34}$$

Denote this fraction by $\mathcal{F}(\varphi, \mathcal{P}, s)$, and say that this fraction corresponds to the measure $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$. Now we would like to apply Theorem 15 to prove the convergence of this fraction. Show that the convergence (3.10) holds for the family $Q_{t_{i-1}t_i}$ defined as $\big(Q_{t_{i-1}t_i}\tilde{g}\big)(z) = \frac{\int\limits_M e^{-\frac{|y-z-\Delta\varphi(t_i)|^2}{2(t_i-t_{i-1})s}} \tilde{g}(y)\,\lambda_M(dy)}{\int\limits_M e^{-\frac{|y-z-\Delta\varphi(t_i)|^2}{2(t_i-t_{i-1})s}} \lambda_M(dy)}$, and the generators (3.32). Applying Corollary 4, we get

$$\frac{\int\limits_M e^{-\frac{|y-z-\Delta\varphi(\tau)|^2}{2\Delta\tau s}} \tilde{g}(y)\,\lambda_M(dy)}{\int\limits_M e^{-\frac{|y-z-\Delta\varphi(\tau)|^2}{2\Delta\tau s}} \lambda_M(dy)} = \tilde{g}\big(z + \Delta\varphi(\tau)\big) - \frac{\Delta\tau s}{2}\Delta_M \tilde{g}\big(z+\Delta\varphi(\tau)\big) + \Delta\tau R_1(\Delta\tau, z + \Delta\varphi(\tau)). \tag{3.35}$$

As the space $E$, we take the space $C(M, \mathbb{R})$. Denote by $Q_{\tau,\tau+\Delta\tau}$ the operator (acting on the function $\tilde{g}$) on the right-hand side. Subtracting $\tilde{g}(z)$ from the both sides, dividing by $\Delta\tau$ and taking the limit as $\Delta\tau \to 0$, we get

$$\lim_{\Delta t\to 0} \frac{(Q_{\tau,\Delta\tau}\,\tilde{g})(z) - \tilde{g}(z)}{\Delta\tau} = (\varphi'(\tau), \nabla_M \tilde{g}(z))_{\mathbb{R}^m} - \frac{s}{2}\Delta_M \tilde{g}(z)\ .$$

We can check immediately that for the operators (3.32), all the assumption of Theorem 16 are fulfilled. The convergence of the fraction (3.34) holds by Theorem 16. Thus, we have proved that the sequence of measures $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ converges weakly (relative to the family of cylinder functions) to the measure $\mathbb{W}^x_{M,\varphi,s}$.

Suppose the function $g$ be such that there exists a number $\tilde{t} < t$, such that $g(\omega) = \tilde{g}(\omega(\tilde{t}))$. Let the number $n$ be such that $\tilde{t} \in (t_n, t_{n+1})$, where $t_n$, $t_{n+1}$ are partition points of $\mathcal{P}_1$. From the previous argument, it follows that we need not to be interested in partition points after the point $t_{n+1}$. Since $\tilde{t}$ is a fixed point of the interval $[t_n, t_{n+1}]$, it is rather convenient to perform all calculations on the interval $[t_n, t_{n+1}]$, but not on the interval $[0, t_{n+1} - t_n]$, as we did when the point $\tilde{t}$ was one of the partition points of $\mathcal{P}_1$. Let $\psi^n(t) = \varphi(t) - \varphi(t_n)$ be a function defined on $[t_n, t_{n+1}]$. We have

$$\int\limits_{C([t_n,t_{n+1}],\mathbb{R}^m)} g(\omega)\, \mathbb{I}_{\{\omega:\, \omega(t_{n+1}) \in U_\varepsilon(M)\}}(\omega)\, \mathbb{W}^z_{\psi^n,s}(d\omega)$$
$$= \int\limits_{C([t_n,t_{n+1}],\mathbb{R}^m)} \tilde{g}(\omega(\tilde{t}) + \psi^n(\tilde{t}))\, \mathbb{I}_{\{\omega:\, \omega(t_{n+1}) \in U_\varepsilon(M)\}}(\omega(t_{n+1}) + \psi^n(t_{n+1}))\mathbb{W}^z_{0,s}(d\omega)$$
$$= \int\limits_{C([t_n,t_{n+1}],\mathbb{R}^m)} \tilde{g}_{-\Delta\varphi(\tilde{t})}(\omega(\tilde{t}) + \psi^n(t_{n+1}))\, \mathbb{I}_{\{\omega:\, \omega(t_{n+1}) \in U_\varepsilon(M)\}}(\omega(t_{n+1}) + \psi^n(t_{n+1}))\mathbb{W}^z_{0,s}(d\omega)$$
$$= \int\limits_{C([t_n,t_{n+1}],\mathbb{R}^m)} g_{-\Delta\varphi(\tilde{t})}(\omega)\, \mathbb{I}_{\{\omega:\, \omega(t_{n+1}) \in U_\varepsilon(M)\}}(\omega)\mathbb{W}^{z+\psi^n(t_{n+1})}_{0,s}(d\omega)$$
$$= \int\limits_{\mathbb{R}^m} \tilde{g}_{-\Delta\varphi(\tilde{t})}(x)\mathbb{W}^x_{0,s} \circ \pi^{-1}_{t_{n+1}-\tilde{t}}(U_\varepsilon(M))\mathbb{W}^{z+\Delta\varphi(t_n)}_{0,s} \circ \pi^{-1}_{\tilde{t}-t_n}(dx)$$

Here $\Delta\varphi(t)$ denotes $\varphi(t_{n+1}) - \varphi(t)$, and the index of the function $g$ below denotes the shift by the corresponding element. Note that $\mathbb{W}^{z+\Delta\varphi(t_n)}_{0,s} \circ \pi^{-1}_{\tilde{t}-t_n}$ and $\mathbb{W}^x_{0,s} \circ \pi^{-1}_{t_{n+1}-\tilde{t}}$ are distributions of multiple Brownian motions with the parameter $s$ starting at the points $z + \Delta\varphi(t_n)$ and $x$ respectively. Rewrite the last integral in the following form:

$$\frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}} \frac{1}{(2\pi s(t_{n+1}-\tilde{t}))^{d/2}} \int\limits_{\mathbb{R}^m} \tilde{g}_{-\Delta\varphi(\tilde{t})}(x)\Big(\int\limits_{\mathbb{R}^m} e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}}\, \mathbb{I}_{U_\varepsilon(M)}(y)dy\Big)e^{-\frac{|x-z-\Delta\varphi(t_n)|^2}{2s(\tilde{t}-t_n)}} dx \,. \tag{3.36}$$

Applying the Fibuni theorem, we get

$$\frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}} \frac{1}{(2\pi s(t_{n+1}-\tilde{t}))^{d/2}} \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\Delta\varphi(t_n)|^2}{2s(\tilde{t}-t_n)}} dx \int\limits_{\mathbb{R}^m} \tilde{g}_{-\Delta\varphi(\tilde{t})}(y)e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}} dy \,.$$

Now let the mesh of the partition $\mathcal{P}_1$ be such that $t_{n+1}$ is close to $\tilde{t}$, so that $|g(y) - g(y - \Delta\varphi(\tilde{t}))| < \sigma$ for all $y \in \mathbb{R}^m$ where $\sigma$ is a number fixed in advance. The integral above, one can rewrite in the form

$$\frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}} \frac{1}{(2\pi s(t_{n+1}-\tilde{t}))^{d/2}} \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\Delta\varphi(t_n)|^2}{2s(\tilde{t}-t_n)}} dx \int\limits_{\mathbb{R}^m} \tilde{g}(y)e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}} dy \tag{3.37}$$
$$+ \delta(z, t_n, t_{n+1}) \,,$$

where for $\delta(z,t_n,t_{n+1})$, one has the following estimate:

$$\begin{aligned}
&|\delta(z,t_n,t_{n+1})| \leqslant \\
&l(s,\tilde{t},t_n,t_{n+1}) \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\Delta\varphi(t_n)|^2}{2s(\tilde{t}-t_n)}}dx \int\limits_{\mathbb{R}^m} |\tilde{g}(y)-\tilde{g}(y-\Delta\varphi(\tilde{t}))| e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}}dy < \\
&\sigma \frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}} \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\Delta\varphi(t_n)|^2}{2s(\tilde{t}-t_n)}}dx = \\
&\sigma \frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}} \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\psi^n(\tilde{t})|^2}{2s(\tilde{t}-t_n)}} k(x,\Delta\varphi(\tilde{t}))dx \leqslant \\
&\sigma \sup_{x\in U_\varepsilon(M)} k(x,\Delta\varphi(\tilde{t}))\, \mathbb{W}_{0,s}^{z+\psi^n(\tilde{t})} \circ \pi^{-1}_{\tilde{t}-t_n}(U_\varepsilon(M)) \ .
\end{aligned} \tag{3.38}$$

Here, by $l(s,\tilde{t},t_n,t_{n+1})$, we denoted $\frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}}\frac{1}{(2\pi s(t_{n+1}-\tilde{t}))^{d/2}}$. As before, $\Delta\varphi(\tilde{t}) = \varphi(t_{n+1}) - \varphi(\tilde{t})$, and $k(x,\Delta\varphi(\tilde{t})) = e^{\frac{2(x-z-\psi^n(\tilde{t}),\Delta\varphi(\tilde{t}))-|\Delta\varphi(\tilde{t})|^2}{2s(\tilde{t}-t_n)}}$. It is clear that $\sup_{x\in U_\varepsilon(M)} k(x,\Delta\varphi(\tilde{t})) \to 1$ for $t_{n+1}\to\tilde{t}$ uniformly in $\varepsilon$ which are smaller than some $\varepsilon_0$. Taking into account the notations introduced above, the integral (3.37) has the form

$$\frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}}\frac{1}{(2\pi s(t_{n+1}-\tilde{t}))^{d/2}} \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\psi^n(\tilde{t})|^2}{2s(\tilde{t}-t_n)}} k(x,\Delta\varphi(\tilde{t}))dx \int\limits_{\mathbb{R}^m} \tilde{g}(y) e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}}dy \ , \tag{3.39}$$

and by the compactness of the set $U_\varepsilon(M)$, uniformly in $\varepsilon < \varepsilon_0$, uniformly in $x \in U_\varepsilon(M)$, the inequality $|k(x,\Delta\varphi(\tilde{t}))-1| < \sigma$ holds for a $\sigma$ fixed in advance, in case if $t_{n+1}$ is close enough to $\tilde{t}$. The absolute value of the difference between the integral (3.39) and the integral

$$\frac{1}{(2\pi s(\tilde{t}-t_n))^{d/2}}\frac{1}{(2\pi s(t_{n+1}-\tilde{t}))^{d/2}} \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\psi^n(\tilde{t})|^2}{2s(\tilde{t}-t_n)}} \int\limits_{\mathbb{R}^m} \tilde{g}(y) e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}}dy \tag{3.40}$$

can be estimated from above by the following value

$$\begin{aligned}
&\sigma\, l(s,\tilde{t},t_n,t_{n+1}) \int\limits_{U_\varepsilon(M)} e^{-\frac{|x-z-\psi^n(\tilde{t})|^2}{2s(\tilde{t}-t_n)}} \int\limits_{\mathbb{R}^m} |\tilde{g}(y)| e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde{t})}}dy \\
&\leqslant K\sigma\, \mathbb{W}_{0,s}^{z+\psi^n(\tilde{t})} \circ \pi^{-1}_{\tilde{t}-t_n}(U_\varepsilon(M)) \ ,
\end{aligned} \tag{3.41}$$

where the constant $K$ is such that $\sup_{y\in U_\varepsilon(M)} |\tilde{g}(y)| \leqslant K$. The last estimate obviously holds uniformly in $\varepsilon$ which are smaller than some $\varepsilon_0$. Consider the integral (3.40). We have

$$\frac{1}{(2\pi s(t_{n+1}-\tilde t))^{d/2}}\int\limits_{\mathbb{R}^m}\tilde g(y)e^{-\frac{|y-x|^2}{2s(t_{n+1}-\tilde t)}}dy=\tilde g(x)-\frac{s(t_{n+1}-\tilde t)}{2}\,\Delta g(x)+O\big((s(t_{n+1}-\tilde t))^{3/2}\big).$$

Taking into account the last relation, and the estimates (3.38) and (3.41), we obtain that the absolute value of the difference between the integral (3.36) and the integral

$$\frac{1}{(2\pi s(\tilde t-t_n))^{d/2}}\int\limits_{U_\varepsilon(M)}e^{-\frac{|x-z-\psi^n(\tilde t)|^2}{2s(\tilde t-t_n)}}\tilde g(x)dx$$

can be estimated by the value $\sigma\,\mathbb{W}_{0,s}^{z+\psi^n(\tilde t)}\circ\pi_{\tilde t-t_n}^{-1}(U_\varepsilon(M))$, where $\sigma$ is a number fixed in advance. Thus,

$$\int\limits_{C([t_n,t_{n+1}],\mathbb{R}^m)}g(\omega)\,\mathbb{W}^z_{M,\psi^n,s,\tilde t-t_n}(d\omega)$$

$$=\lim_{\varepsilon\to0}\frac{\int\limits_{U_\varepsilon(M)}\tilde g(x)\mathbb{W}_{0,s}^{z+\psi^n(\tilde t)}\circ\pi_{\tilde t-t_n}^{-1}(dx)+\delta(\tilde t-t_n,z,\psi^n)\mathbb{W}_{0,s}^{z+\psi^n(\tilde t)}\circ\pi_{\tilde t-t_n}^{-1}(U_\varepsilon(M))}{\mathbb{W}_{0,s}^{z+\psi^n(\tilde t)}\circ\pi_{\tilde t-t_n}^{-1}\big(U_\varepsilon(M)\big)}$$

$$=\frac{\int\limits_M e^{-\frac{|x-z-\psi^n(\tilde t)|^2}{2(\tilde t-t_n)s}}\tilde g(x)\,\lambda_M(dx)}{\int\limits_M e^{-\frac{|x-z-\psi^n(\tilde t)|^2}{2(\tilde t-t_n)s}}\lambda_M(dx)}+\delta(\tilde t-t_n,z,\psi^n)\ ,$$

where $|\delta(\tilde t-t_n,z,\psi^n)|<\sigma$. For convenience of further applications, we rewrite this equality in different notations:

$$\int\limits_{C([0,t_{n+1}-t_n],\mathbb{R}^m)}g(\omega)\,\mathbb{W}^z_{M,\varphi_{t_nt_{n+1}},s,\tilde t-t_n}(d\omega)$$
$$=\frac{\int\limits_M e^{-\frac{|x-z-\varphi_{t_nt_{n+1}}(\tilde t-t_n)|^2}{2(\tilde t-t_n)s}}\tilde g(x)\,\lambda_M(dx)}{\int\limits_M e^{-\frac{|x-z-\varphi_{t_nt_{n+1}}(\tilde t-t_n)|^2}{2(\tilde t-t_n)s}}\lambda_M(dx)}+\delta(\tilde t-t_n,z,\varphi_{t_nt_{n+1}}(\tilde t-t_n)) \tag{3.42}$$

where $|\delta(\tilde t-t_n,z,\varphi_{t_nt_{n+1}}(\tilde t-t_n))|<\sigma$. Let us remind that $\mathcal{F}(\varphi,\mathcal{P},s)$ is the notation for the fraction (3.34). Thus, for the case which we consider, the integral (3.31) can be presented as

$$\int\limits_{C([0,1],\mathbb{R}^m)}h(\omega)\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega)=\mathcal{F}(\varphi,\mathcal{P},s)+\delta(x,\tilde t-t_n)\ ,$$

where $|\delta(x,\tilde{t}-t_n)|<\sigma$. □

## 3.5 The second step of construction of $\mathbf{W}_M^x$

Define the measure $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ by the formula

$$\int\limits_{C([0,s],\mathbb{R}^m)^1} f(\omega)\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,s],\mathbb{R}^m)^{t_1}} \tilde{\mathbb{W}}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int\limits_{C([0,s],\mathbb{R}^m)^{t_2-t_1}} \tilde{\mathbb{W}}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2)\ldots \int\limits_{C([0,s],\mathbb{R}^m)^{t_n-t_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) f(\omega_1,\omega_2,\ldots,\omega_n)\ . \tag{3.43}$$

The measure $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ is well defined. To verify this, we have to show that each function $\varphi_{t_{i-1}t_i}$ is such that $\varphi_{t_{i-1}t_i}(0)=0$, which obviously holds, and we have to show that each function $\omega_i(t_i-t_{i-1})$ is such that $\omega_i(t_i-t_{i-1})(0)\in M$. Verify the last assertion. We have

$$\begin{gathered}\omega_1(t_1)(0)=\omega(t_1)(0)=x+\varphi(t_1)\in M;\\ \omega_2(t_2-t_1)(0)=\omega_1(t_1)(0)+\varphi_{t_1t_2}(t_2-t_1)\\ =x+\varphi(t_1)+\varphi(t_2)-\varphi(t_1)=x+\varphi(t_2)\in M;\\ \cdots\\ \omega_i(t_i-t_{i-1})(0)=\omega_{i-1}(t_{i-1}-t_{i-2})(0)+\varphi_{t_it_{i-1}}(t_i-t_{i-1})=x+\varphi(t_i)\in M.\end{gathered}$$

Now we changes the roles of the variables $s$ and $t$, so that $s$ is the time variable, and $t$ is the variable of functions from the state space. Since $s$ is the time parameter, then we rewrite the process $\mathbf{W}^x_{M,\varphi,s}$ as $\mathbf{W}^{x+\varphi}_{M,s}$. The function $x+\varphi$ is the value of the process at the moment 0. In the following, it will be convenient to consider that $\varphi(0)=x$, and to denote the process discussed above by $\mathbf{W}^{\varphi}_{M,s}$. The distribution of the process $\mathbf{W}^{\varphi}_{M,s}$, we denote by $\mathbb{W}^{\varphi}_{M,s}$.

Let $\mathcal{P}_2=\{s_0=0<s_1<\ldots<s_k=1\}$ be a partition of the interval $[0,1]$. In the following construction, the time parameter is the parameter $s$. Define the measure $\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ by the formula

$$\int\limits_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega) = \int\limits_{C([0,1],\mathbb{R}^m)^{s_1}} \tilde{\mathbb{W}}^x_{M,s_1,\mathcal{P}_1}(d\omega_1) \int\limits_{C([0,1],\mathbb{R}^m)^{s_2-s_1}} \tilde{\mathbb{W}}^{\omega_1(s_1)}_{M,s_2-s_1,\mathcal{P}_1}(d\omega_2)\ldots \int\limits_{C([0,1],\mathbb{R}^m)^{s_n-s_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(s_{n-1}-s_{n-2})}_{M,s_n-s_{n-1},\mathcal{P}_1}(d\omega_n) f(\omega_1,\ldots,\omega_n)\ .$$

THEOREM 18 (On a Brownian sheet). *For every $x \in M$, as $|\mathcal{P}_1| \to 0$ and $|\mathcal{P}_2| \to 0$, $\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ converges weakly relative to the family of bounded continuous cylinder functions to a measure $\mathbb{W}^x_M$. The measure $\mathbb{W}^x_M$ has the following properties:*

(i) *considered as the distribution of a $\mathrm{C}([0,1],M)$-valued process, $\mathbb{W}^x_M$ possesses a transition probability at time $t$ which is the distribution of a Brownian motion on $M$ with variance $t$ starting at the point $x$;*

(ii) *considered as the distribution of an $M$-valued two-parameter stochastic process, $\mathbb{W}^x_M$ possesses a transition probability at time $(s,t)$ which is the heat kernel measure on $M$ at time $st$, i.e., if $f \in \mathrm{C}(M)$, then*

$$\mathbb{W}^x_M \circ \pi^{-1}_{s,t}\,[f] = \big(e^{-\frac{st}{2}\Delta_M} f\big)(x),$$

*where $\pi_{s,t}$ is the mapping $\mathrm{C}([0,1]\times[0,1],M) \to M$, $\omega \mapsto \omega(s,t)$, $\Delta_M$ is the Laplace–Beltrami operator on $M$, $e^{-\frac{\tau}{2}\Delta_M}$ is the heat semigroup.*

*Proof.* It suffices to prove the convergence in case when the function $f$ is such that $f(\omega) = \tilde{f}(\omega(s))$ for some function $\tilde{f} : C([0,1],\mathbb{R}^m) \to \mathbb{R}$. We have

$$\int\limits_{C([0,s],\mathbb{R}^m)^1} f(\omega)\,\tilde{\mathbb{W}}^{\varphi}_{M,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{f}(w)\,\tilde{\mathbb{W}}^{\varphi}_{M,s,\mathcal{P}_1} \circ \pi_s^{-1}(dw) = \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{f}(w)\,\mathbb{W}^{\varphi}_{M,s,\mathcal{P}_1}(dw).$$

From this, it follows that

$$\begin{aligned}
\int\limits_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega) &= \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{\mathbb{W}}^x_{M,s_1,\mathcal{P}_1} \circ \pi^{-1}_{s_1}(dw_1) \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{\mathbb{W}}^{w_1}_{M,s_2-s_1,\mathcal{P}_1} \circ \pi^{-1}_{s_2-s_1}(dw_2)\dots \\
&\quad \int\limits_{C([0,1],\mathbb{R}^m)} \tilde{\mathbb{W}}^{w_{n-1}}_{M,s_n-s_{n-1},\mathcal{P}_1} \circ \pi^{-1}_{s_n-s_{n-1}}(dw_n)\tilde{f}(w_n) \\
&= \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^x_{M,s_1,\mathcal{P}_1}(dw_1) \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_1}_{M,s_2-s_1,\mathcal{P}_1}(dw_2)\dots \\
&\quad \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_{n-2}}_{M,s_{n-1}-s_{n-2},\mathcal{P}_1}(dw_{n-1}) \int\limits_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_{n-1}}_{M,s_n-s_{n-1},\mathcal{P}_1}(dw_n)\tilde{f}(w_n).
\end{aligned}$$

Suppose first that the function $f$ be such that there exist a function $p : \mathbb{R}^m \to \mathbb{R}$ and numbers $t, s \in [0,1]$, such that $f(\omega) = p(\omega(t,s))$. Taking into account that each of these integrals has the form (3.34), we obtain that the integral

$\int_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)$ has the form:

$$\frac{\int_M e^{-\frac{|x_1-x|^2}{2\Delta s_1\Delta t_1}}dx_1}{\int_M e^{-\frac{|\bar{x}_1-x|^2}{2\Delta s_1\Delta t_1}}d\bar{x}_1}\cdots\frac{\int_M e^{-\frac{|x_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}}dx_{n-1}}{\int_M e^{-\frac{|\bar{x}_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}}d\bar{x}_{n-1}}\frac{\int_M e^{-\frac{|x_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}}dx_n}{\int_M e^{-\frac{|\bar{x}_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}}d\bar{x}_n}$$

$$\frac{\int_M e^{-\frac{|y_1-x_1|^2}{2\Delta s_2\Delta t_1}}dy_1}{\int_M e^{-\frac{|\bar{y}_1-x_1|^2}{2\Delta s_2\Delta t_1}}d\bar{y}_1}\cdots\frac{\int_M e^{-\frac{|y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}}dy_{n-1}}{\int_M e^{-\frac{|\bar{y}_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}}d\bar{y}_{n-1}}\frac{\int_M e^{-\frac{|y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}}dy_n}{\int_M e^{-\frac{|\bar{y}_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}}d\bar{y}_n}$$

$$\cdots$$

$$\frac{\int_M e^{-\frac{|u_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}}du_1}{\int_M e^{-\frac{|\bar{u}_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}}d\bar{u}_1}\cdots\frac{\int_M e^{-\frac{|u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}}du_{n-1}}{\int_M e^{-\frac{|\bar{u}_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}}d\bar{u}_{n-1}}\frac{\int_M e^{-\frac{|u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}}du_n}{\int_M e^{-\frac{|\bar{u}_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}}d\bar{u}_n}$$

$$\frac{\int_M e^{-\frac{|v_1-u_1|^2}{2\Delta s_k\Delta t_1}}dv_1}{\int_M e^{-\frac{|\bar{v}_1-u_1|^2}{2\Delta s_k\Delta t_1}}d\bar{v}_1}\cdots\frac{\int_M e^{-\frac{|v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}}dv_{n-1}}{\int_M e^{-\frac{|\bar{v}_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}}d\bar{v}_{n-1}}\frac{\int_M e^{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}p(v_n)dv_n}{\int_M e^{-\frac{|\bar{v}_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}d\bar{v}_n},$$

where $\Delta t_i = t_i - t_{i-1}$, $\Delta s_j = s_j - s_{j-1}$. Denote this integral by $I(\mathcal{P}_1,\mathcal{P}_2,p)$. Let the constant $K$ be such that $\|p\| \leqslant K$. Then all the integrals in $I(\mathcal{P}_1,\mathcal{P}_2,f)$, and the integral $I(\mathcal{P}_1,\mathcal{P}_2,f)$ itself are bounded by the same constant. Without loss of generality, we can set $\lambda_M(M) = 1$.

LEMMA 26. *$I(\mathcal{P}_1,\mathcal{P}_2,p)$ converges to $e^{-\frac{st}{2}\Delta_M}p$, if the meshes of $|\mathcal{P}_1|$ and $\mathcal{P}_2$ tend to zero.*

*Proof.* First, we prove the lemma for those functions $p$ for which $e^{-\frac{t}{2}\Delta_M}p$ can be presented as a convergent exponential series for all $t \geqslant 0$. According to the book [13], the set of such vectors is dense in $C(M,\mathbb{R})$. If lemma is proved for such vectors, then, by the Banach-Steinhaus theorem, the statement holds for all $p \in C(M,\mathbb{R})$.

First, we consider the last line of $I(\mathcal{P}_1,\mathcal{P}_2,p)$ for $u_1, \ldots, u_n$, satisfying the condition $|u_i - u_{i-1}| < (|\mathcal{P}_2|\Delta t_i)^\alpha$, $\alpha > 0$. We apply the asymptotic (3.25) from Proposition 4. The number $N$, we choose so that $(|\mathcal{P}_2|)^{\alpha N} < \Delta s_k|\mathcal{P}_2|^\alpha$. Show, for example, how it looks for the last right integral.

$$\begin{aligned}\frac{\int_M e^{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}p(v_n)dv_n}{\int_M e^{-\frac{|\bar{v}_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}d\bar{v}_n} &= p(v_{n-1}+(u_n-u_{n-1})_M^{v_{n-1}})\\ &\quad-\frac{\Delta s_k\Delta t_n}{2}\Delta_M p(v_{n-1}+(u_n-u_{n-1})_M^{v_{n-1}})\\ &\quad+\mathcal{R}(v_{n-1},u_n-u_{n-1})\\ &\quad+\Delta s_k\Delta t_n R_2(\Delta s_k\Delta t_n,|\mathcal{P}_2|,v_{n-1},u_n-u_{n-1}),\end{aligned}$$

Further, we integrate only first three terms in this expression, since by Proposition 4, $|R_2(\Delta s_k \Delta t_n, |\mathcal{P}_2|, v_{n-1}, u_n - u_{n-1})| < K(\mathcal{P}_2 \Delta t_n)^\alpha$, where $K$ is a constant. At each term of the form $(\Delta s_k)^l \Delta t_{i_1} \dots \Delta t_{i_l}$, $i_1 > \dots > i_l$, we determine the coefficient depending on $u_1, \dots, u_n$, which appears after applying the asymptotic (3.25) from Proposition 4 to each integral of the of the last line in $I(\mathcal{P}_1, \mathcal{P}_2, p)$. For convenience of writing of the coefficients, we introduce the following notations. Let $y \in M$.

$$\begin{aligned}
&y_1 = y, \\
&y_2 = y_1 + (u_{i+1} - u_i)_M^{y_1}, \\
&y_3 = y_2 + (u_{i+2} - u_{i+1})_M^{y_2}, \\
&\dots \\
&y_{n-i+1} = y_{n-i} + (u_n - u_{n-1})_M^{y_{n-i}}.
\end{aligned}$$

Further, if $g$ is a function on $M$, then define $g_{(u_n u_{i+1})_M}(y)$ by the formula

$$g_{(u_n u_{i+1})_M}(y) = g(y_{n-i+1}).$$

First, consider the terms of an order not exceeding the order one.

$$p(u_n) - \frac{\Delta s_k \Delta t_n}{2} \Delta_M p(u_n) - \sum_{i=1}^{n-1} \frac{\Delta s_k \Delta t_{n-i}}{2} \Delta_M p_{(u_n u_{n-i+1})_M}(u_{n-i}),$$

Then, we consider the coefficient at the term of the form $(\Delta s_k)^l \Delta t_{i_1} \dots \Delta t_{i_l}$, $i_1 > \dots > i_l$:

$$\frac{1}{2^l} \Delta_M \Big( \dots \Big( \Delta_M p_{(u_n u_{i_1+1})_M} \Big)_{(u_{i_1} u_{i_2+1})_M} \dots \Big)_{(u_{i_{l-1}} u_{i_l+1})_M} (u_{i_l}) \tag{3.44}$$

Now, we have determined the terms which appeared as a result of presence of terms of the form $\mathcal{R}(v_{n-1}, u_n - u_{n-1})$ in the asymptotic which was applied. Note that the terms containing summands of such form, or products of expressions containing other expressions of such form, after applying of the integrals of the first line, contain the expression of the form $(u_{i+1} - u_i)_\perp^{u_i}$, which is equal to zero. Further, present $R(v_{n-1}, u_n - u_{n-1})$ in the form

$$\begin{aligned}
\mathcal{R}(v_{n-1}, u_n - u_{n-1}) = (\Delta_M \imath(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}), (u_n - u_{n-1})_\perp^{v_{n-1}})^2 \\
+ \mathcal{R}_3(v_{n-1}, u_n - u_{n-1}),
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{R}_3(v_{n-1}, u_n - u_{n-1}) \\
&= \sum_{n=3}^{N} \sum k(n) \prod_{l_1 + \dots + l_s = n} D_M^{2,l_i}(\imath(\cdot), (u_n - u_{n-1})_\perp^{v_{n-1}})^{l_i} \big(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}\big),
\end{aligned}$$

Applying the operator $\Delta_M$ to $\mathcal{R}_3(v_{n-1}, u_n - u_{n-1})$ always produces the expression $(u_n - u_{n-1})_\perp^{v_{n-1}}$, which has the order $|\mathcal{P}_2|^\alpha$. Besides that, the operator $\Delta_M$ appears with the factor $\Delta s_k \Delta t_{n-1}$. Such a term will be included in the rest term, which can be estimated from above by $K\Delta s_k \Delta t_{n-1}|\mathcal{P}_2|^\alpha$, where $K$ is a constant. Applying the asymptotic to the second integral from the end, produces the term $\mathcal{R}_3(v_{n-2}, u_{n-1} - u_{n-2})$. The next applying of the operator $\Delta_M$ to this term, produces another term, which can be analogously estimated from above by $K\Delta s_k \Delta t_{n-2}|\mathcal{P}_2|^\alpha$, and so on.

Prove that

$$|\Delta_M(\Delta_M \imath(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}), (u_n - u_{n-1})_\perp^{v_{n-1}})^2| < K|u_n - u_{n-1}|,$$

where the first from the left operator $\Delta_M$ is applied to a function of variable $v_{n-1}$, $K$ is a constant. First, consider the case $v_{n-1} \neq u_{n-1}$. Calculate $\Delta_M(u_n - u_{n-1})_\perp^{v_{n-1}}$ differentiating in $v_{n-1}$. At the point $v_{n-1}$, we consider geodesics $\gamma_i(t)$, such that $\gamma_i'(0)$ form an orthonormal basis in the space which is tangent to $M$ at the point $v_{n-1}$. To show that

$$\left|\frac{d^2}{dt^2}(\Delta_M \imath\big(\gamma_i(t) + (u_n - u_{n-1})_M^{\gamma_i(t)}), (u_n - u_{n-1})_\perp^{\gamma_i(t)}\big)^2\Big|_{t=0}\right| < K|u_n - u_{n-1}|,$$

it suffices to show that

$$|\frac{d}{dt}(u_n - u_{n-1})_\perp^{\gamma_i(t)}\Big|_{t=0}| < K|u_n - u_{n-1}| \ . \tag{3.45}$$

Indeed, it is easy to notice that in case when $v_{n-1} \neq u_{n-1}$, $(u_n - u_{n-1})_\perp^{v_{n-1}}$ has the form

$$(u_n - u_{n-1})_\perp^{\gamma_i(t)} = k(t)|u_n - u_{n-1}|n_M(\gamma_i(t)),$$

where $k(t)$ is a scalar function not exceeding 1 and having a bounded derivative, $n_M(x)$ is a unit vector from the space normal to $M$ at the point $x$. From this, it follows that the derivative at zero is proportional to $|u_n - u_{n-1}|$. Further, we prove that

$$|\Delta_M(\Delta_M \imath(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}), (u_n - u_{n-1})_\perp^{v_{n-1}})^2|_{v_{n-1}=u_{n-1}}| < K|u_n - u_{n-1}|.$$

Consider tangent spaces $T_{u_{n-1}}$ and $T_{u_n}$ at the points $u_{n-1}$ and $u_n$, respectively. In each of these spaces, we construct a space which is parallel to the space of intersection of $T_{u_{n-1}}$ $T_{u_n}$. These subspaces have codimension 1 $T_{u_{n-1}}$ and $T_{u_n}$, respectively.

Denote these subspaces by $T'_{u_{n-1}}$ and $T'_{u_n}$, respectively. At the points $u_{n-1}$ and $u_n$, we choose geodesics $\gamma_i^{n-1}$ and $\gamma_i^n$, such that the vectors of their derivatives at zero, form an orthonormal basis in the spaces $T_{u_{n-1}}$ and $T_{u_n}$, respectively, moreover, the first $d-1$ vectors of those bases belong to the spaces $T'_{u_{n-1}}$ and

$T'_{u_n}$, respectively (the tangent spaces have the dimension $d$). As before, we have to prove (3.45). Consider a two-dimensional curve which is the intersection of the manifold $M$ with the plane spanned on the vectors $(\gamma_i^{n-1})'(0)$ and $u_n - u_{n-1}$. From this, we see that $\frac{d}{dt}(u_n - u_{n-1})_{\perp}^{\gamma_i^{n-1}(t)}\big|_{t=0} = 0$ for $i = 1, \ldots, d-1$. The vector $e_d = (\gamma_d^{n-1})'(0)$ is orthogonal to $T'_{u_{n-1}}$. Let $P$ be the point of intersection of the line $u_{n-1} + te_d$ with $T_{u_{n-1}} \cap T_{u_n}$. Connect the points $P$ and $u_n$. We got a triangle, where the sines of the the acute angles are proportional to $u_n - u_{n-1}$. From the geometrical consideration, it follows that $|(u_n - u_{n-1})_{\perp}^{\gamma_i(t)}| < K\,t\,|u_n - u_{n-1}|$, $K$ is a constant. This proves (3.45).

Thus, under the assumption $|u_{i+1} - u_i| < (\Delta t_i |\mathcal{P}_2|)^{\alpha}$, we found the general term of the series, which we obtained after applying the last line of the integral $I(p, \mathcal{P}_1, \mathcal{P}_2)$ (the coefficient (3.44), multiplied by $(\Delta s_k)^l \Delta t_{i_1} \ldots \operatorname{var} t_{i_l}$, $i_1 > \cdots > i_l$), and we proved the the rest term does not exceed

$$K\,t\,\Delta s_k\,|\mathcal{P}_2|^{\alpha} |\mathcal{P}_1|^{\alpha}\ , \tag{3.46}$$

where $K$ is a constant.

Starting from the integrals of the first line in the integral $I(p, \mathcal{P}_1, \mathcal{P}_2)$, we replace integration over the manifold with the integration over sufficiently small neighborhoods on the manifold. Then, we pass to the iterative limit: first to the limit as $|\mathcal{P}_1| \to 0$, and then to the limit as $|\mathcal{P}_2| \to 0$. Describe the choice of the neighborhoods in detail. Choose $0 < \alpha < \tilde{\alpha} < \frac{1}{2}$, $\Delta\alpha = \tilde{\alpha} - \alpha$. In the integral of the form $\frac{\int_M e^{-\frac{|x_i - x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} h(x_i) dx_i}{\int_M e^{-\frac{|\bar{x}_i - x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} d\bar{x}_i}$, on the $i$th position, we replace the integration over the whole manifold with the integration over the neighborhood $U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde{\alpha}} (\Delta t_i)^{\tilde{\alpha}})$ of the point $x_{i-1}$ of radius $|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}}$. Here, $h$ denotes a general form of function to which we apply the above integration operator (in our case, this function is a sequence of integrals applied to the function $p$).

We have

$$\frac{1}{(2\pi\Delta s_1 \Delta t_i)^{\frac{d}{2}}} \int\limits_{M \setminus U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}})} e^{-\frac{|x_i - x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} h(x_i)\,dx_i < \frac{C}{(2\pi\Delta s_1 \Delta t_i)^{\frac{d}{2}}}\, e^{-\frac{1}{(\Delta s_1 \Delta t_i)^{1-2\tilde{\alpha}}}},$$

where $C$ is such that $|p(y)| < C$ for all $y \in M$ (and hence, all subintegral functions of each integral in $I(p, \mathcal{P}_1, \mathcal{P}_2)$, are bounded with the same constant). As it was mentioned above, $\lambda(M) = 1$, which we can accept without loss of generality. By Corollary 3, if the mesh of the partition $\mathcal{P}_1$ is small enough, we obtain

$$\frac{\int\limits_{M \setminus U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}})} e^{-\frac{|x_i - x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} h(x_i)\,dx_i}{\int_M e^{-\frac{|\bar{x}_i - x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} d\bar{x}_i} < \frac{C}{(2\pi\Delta s_1 \Delta t_i)^{\frac{d}{2}}}\, e^{-\frac{1}{(\Delta s_1 \Delta t_i)^{1-2\tilde{\alpha}}}}$$

Thus, each $i$th integral of the first line, we replace with an integral of the form

$$\frac{\int\limits_{U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha})} e^{-\frac{|x_i-x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} h(x_i)\,dx_i}{\int_M e^{-\frac{|\bar x_i - x_{i-1}|^2}{2\Delta s_1 \Delta t_i}} d\bar x_i},$$

the form of the function $h$ is described above.

Further,for the $i$th integral of the second line, we have $|x_i - x_{i-1}| < |\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}$. Fix a neighborhood $U_{y_{i-1}}(2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}) \subset M$ of the point $y_{i-1}$, so that for all points $y_i$ of which, the inequality $|y_i - y_{i-1}| < 2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}$ holds. We have

$$|y_i - y_{i-1} - (x_i - x_{i-1})| > |y_i - y_{i-1}| - |x_i - x_{i-1}| > (2^{\Delta\alpha} - 1)|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha} .$$

Taking into account this inequality, Proposition 4, analogously to the estimate obtained above, we get

$$\frac{\int_{M\setminus U_{y_{i-1}}(2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha})} e^{-\frac{|y_i-y_{i-1}-x_i+x_{i-1}|^2}{2\Delta s_2 \Delta t_i}} h(y_i)\,dy_i}{\int_M e^{-\frac{|\bar y_i-y_{i-1}-x_i+x_{i-1}|^2}{2\Delta s_2 \Delta t_i}} d\bar y_i} < \frac{C}{(2\pi\Delta s_2 \Delta t_i)^{\frac{d}{2}}}\, e^{-\frac{(2^{\Delta\alpha}-1)^{\tilde\alpha}}{(\Delta s_2 \Delta t_i)^{1-2\tilde\alpha}}} .$$

Continue choosing neighborhoods in the described way, so that we consider the $i$th integral of the last line for points for the points $u_i$ and $u_{i-1}$ such that $|u_i - u_{i-1}| < (k-1)^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha} < |\mathcal{P}_2|^{\alpha}(\Delta t_i)^{\alpha}$, where the last inequality holds for a sufficiently small mesh of the partition $|\mathcal{P}_1|$, which satisfies $(k|\mathcal{P}_2|\Delta t_i)^{\Delta\alpha} < 1$.

Further, starting from the last integral in the last line, we apply the asymptotic (3.25) from Proposition 4, as we described above.

Now, we consider the second, from the end, line of integral operators. As before, for a sufficiently small mesh of the partition $\mathcal{P}_1$, we can consider that $|z_i - z_{i-1}| < (\Delta t_i\,|\mathcal{P}_2|)^{\alpha})$ for all $i$. We act with the sequence of integral operators on each term produced after obtaining the asymptotic in the last line, besides the term which we denoted as the rest term. Here, we apply the asymptotic from Proposition 4. Applying the first $k-1$ lines to the rest term, gives us a new rest term, which can be also estimated by (3.46). Therefore, we exclude the rest term from our consideration. Next, we act with the third, counting from the end, line of integral operators on all terms obtained produced from the last two lines, and so on. At the end, we act Thus, we get an asymptotic expression for the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$. Write down the general term of this asymptotic expression (up to the sign):

$$\frac{(\Delta s_k)^{l_0}\Delta t_{i_1}\cdots\Delta t_{i_{l_0}}(\Delta s_{k_1})^{l_1}\Delta t_{j_1}\cdots\Delta t_{j_{l_1}}\cdots(\Delta s_{k_s})^{l_s}\Delta t_{m_1}\cdots\Delta t_{m_{l_s}}}{2^{l_0+l_1+\cdots+l_s}}$$
$$\times\left(\Delta_M^{(y_1^s)}\right)^{l_{s-q}}\cdots\left(\Delta_M^{(y_q^s)}\right)^{l_s}\Big(\cdots\left(\Delta_M^{(y_1^1)}\right)^{l_{r+1}}\cdots\left(\Delta_M^{(y_p^1)}\right)^{l_p}\Big(\left(\Delta_M^{(y_1^0)}\right)^{l_1}\cdots\left(\Delta_M^{(y_r^0)}\right)^{l_r}$$
$$\left(\Delta_M^{l_0}p_{\sum_i(y_i^0-x)_M}(x)\right)_{\sum_i(y_i^1-x)_M}\Big|_{y^0=x}\Big)_{\sum_i(y_i^2-x)_M}\Big|_{y^1=x}\cdots\Big|_{y^{s-1}=x}\Big)\Big|_{y^s=x} \tag{3.47}$$

Here, $l_i \geqslant 0$, $i=1,\dots,s$, $k>k_1>\cdots>k_s$, the index of $(y_i^j)$ above, denotes that the operator $\Delta_M^p$ is applied to the function of variable $y_i^j$, $y^i=x$ denotes that $y_j^i=x$ for all $j$. For an arbitrary function $\varphi$, defined on $M$, $\varphi_{\sum_i(w_i)_M}$ denotes $\varphi_{\sum_i(w_i)_M}(y)=\varphi(y+(w_1)_M^y+(w_2)_M^{y+(w_1)_M^y}+\cdots)$, where $w_i$ are sufficiently small so that $\varphi_{\sum_i(w_i)_M}$ is well defined. Calculating of (3.47) in local coordinates, chosen in a neighborhood of the point $x$, gives

$$\frac{(\Delta s_k)^{l_0}\Delta t_{i_1}\cdots\Delta t_{i_{l_0}}(\Delta s_{k_1})^{l_1}\Delta t_{j_1}\cdots\Delta t_{j_{l_1}}\cdots(\Delta s_{k_s})^{l_s}\Delta t_{m_1}\cdots\Delta t_{m_{l_s}}}{2^{l_0+l_1+\cdots+l_s}}\Delta_M^{l_0+l_1+\cdots+l_s}(x)$$

Compare this expression with the extension for

$$e^{-\frac{st}{2}\Delta_M}=e^{-\frac{\Delta s_1\Delta t_1}{2}\Delta_M}\cdots e^{-\frac{\Delta s_k\Delta t_n}{2}\Delta_M},$$

if we apply the asymptotic $e^{-\frac{\Delta s_i\Delta t_j}{2}\Delta_M}=1-\frac{\Delta s_i\Delta t_j}{2}\Delta_M+O((\Delta s_i\Delta t_j)^{\frac{3}{2}})$ consequently to each exponent starting from the right. Up to the terms which tend to zero as $|\mathcal{P}_1|\to 0$, $|\mathcal{P}_2|\to 0$, the asymptotic extensions will coincide.

By symmetry of the integral $I(\mathcal{P}_1,\mathcal{P}_2,p)$, we can first take the limit in $n$ for a fixed $k$, and then, the limit in $k$. Both iterative limits coincide and equal to $e^{-\frac{st}{2}\Delta_M}$. Hence, there exists a double limit as $|\mathcal{P}_1|\to 0$, $|\mathcal{P}_2|\to 0$, and is equal to $e^{-\frac{st}{2}\Delta_M}$. □

If the function $f$ is such that $f(\omega)=p(\omega(\tilde{t},\tilde{s}))$, where $\tilde{t}<t$, $\tilde{s}<s$, then we use the proof of Theorem 17, from which, it follows that

$$\int_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)=I(p,\mathcal{P}_1,\mathcal{P}_2)+\delta(x,\tilde{t}-t_n,\tilde{s}-s_n)$$

$s_n$ and $t_n$ are partition points of $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\tilde{t}\in(t_n,t_{n+1})$, $\tilde{s}\in(s_n,s_{n+1})$, and if $\sigma$ is an arbitrary sufficiently small number, then for sufficiently small meshes of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, the inequality $|\delta(x,(\tilde{t}-t_n,\tilde{s}-s_n))|<\sigma$ holds. For the function $f$ depending on $\omega$ in several points, say in the points $\xi_i\in[0,s]$ and $\tau_j\in[0,t]$, the form of the integral $I(\mathcal{P}_1,\mathcal{P}_2,p)$ will be the same, and the convergence can be proved analogously. The integral converges to the product of of operators of the form $e^{-\frac{\Delta\xi_i\Delta\tau_j}{2}\Delta_M}$, each of these operators acts on the corresponding variable of the function $p$, which is defined as $f(\omega) = p(\omega_{11}(\xi_1,\tau_1),\dots,\omega_{kl}(\xi_l,\tau_l))$, where $\omega_{ij}$ is defined on $[0,\xi_i-\xi_{i-1}]\times[0,\tau_j-\tau_{j-1}]$ by the formula $\omega_{ij}(s,t)=\omega(\xi_{i-1}+s,\tau_{j-1}+t)$. The theorem is proved. □

Corollary 5. *Let $M$ be a compact Lie group. Then, $\mathbf{W}_M^x$ considered as a $C([0,1],M)$-valued process, coincides with the Brownian motion constructed by Malliavin in [18].*

*Proof.* The proof follows from Theorem 17 and Theorem 2.15 from the paper [12]. Indeed, Theorem 2.15 from the paper [12] and Theorem 17 imply the coincidence of finite dimensional distributions. Indeed, by Theorem 2.15 from [12], for the process constructed in [18] the following statement holds: for each fixed $s$, the process is a Brownian motion on $M$ with the parameter $s$. The coincidence of finite dimensional distributions, follows now from a formula in the book [15] on the page 204. Thus, the distributions of the processes coincide on the algebra of cylindric sets of the space $C([0,1],(C([0,1],M))$, where in $C([0,1],M)$, we also fix the algebra of cylindric sets. This implies the $\sigma$-additivity of the measure $\mathbb{W}_M^x$, since the distribution of a Brownian motion, constructed in [18], is a $\sigma$-additive measure. Thus, $\mathbb{W}_M^x$ is defined on the $\sigma$-algebra of Borel sets of the space $C([0,1],C([0,1],M))$, $\sigma$-additive, and coincides with the distribution of a Brownian motion from the paper [18]. □

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