

# Constructing a Brownian Sheet with Values in a Compact Riemannian Manifold

É. Yu. Shamarova

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In this paper, we propose a new method of constructing a two-parameter random field  $\mathbf{W}_M^x(s, t)$ ,  $x \in M$ , with values in a compact Riemannian manifold  $M$  possessing the property that the random processes  $\mathbf{W}_M^x(\cdot, t)$  and  $\mathbf{W}_M^x(s, \cdot)$  are Brownian motions on the manifold  $M$  with parameters  $t$  and  $s$ , respectively, issuing from the point  $x$ . (By a *Brownian motion* on a manifold  $M$  with parameter  $t$  we mean the diffusion process generated by the operator  $-(t/2)\Delta_M$ , where  $\Delta_M$  is the Laplace operator on the manifold  $M$ .) For the case in which the manifold is a compact Lie group, the two-parameter random field constructed in the paper coincides with the Brownian sheet defined by Malliavin [1] in 1991. (Malliavin called this random field a Brownian motion with values in  $C([0, 1], M)$ , which is the set of continuous functions defined on the closed interval  $[0, 1]$  and taking values in  $M$ .) Nevertheless, for the case in which the manifold is a compact Lie group, the method proposed in the present paper essentially differs from that used in Malliavin's paper.

## 1. FIRST STEP IN THE CONSTRUCTION OF THE RANDOM FIELD $\mathbf{W}_M^x$

Suppose that  $M$  is a  $d$ -dimensional compact Riemannian manifold without boundary isometrically embedded in  $\mathbb{R}^m$ . By a *Brownian sheet with values in  $\mathbb{R}^m$*  we mean the family of  $m$  independent standard Brownian sheets. Suppose that  $\mathbf{W}_{t,s}$  is an  $n$ -dimensional Brownian sheet. Consider  $\mathbf{W}_{t,s}$  as a process taking values in the space  $C([0, 1], \mathbb{R}^m)$ . We denote this process by the symbol  $\mathbf{W}_t$ . We introduce the following notation: if  $E$  is a locally convex space, then  $E^t$  denotes  $C([0, t], E)$ ; if  $y \in C([0, 1], \mathbb{R}^m)$  is a continuous function, then  $\mathbb{W}^y$  denotes the distribution of the process  $\mathbf{W}_t^y = y + \mathbf{W}_t$ . If  $\psi \in C([0, 1], \mathbb{R}^m)$ , then we define the process  $(\mathbf{W}_\psi^y)_t = \psi(t) + \mathbf{W}_t^y$ . Suppose that  $\widetilde{\mathbb{W}}_\psi^y$  is the distribution of this process and  $\mathbf{E}_{y,\psi}$  is the expectation with respect to the measure  $\widetilde{\mathbb{W}}_\psi^y$ . Further,  $U_\varepsilon(M)$  denotes the  $\varepsilon$ -neighborhood of the manifold  $M$ . We consider  $\mathbf{W}_\psi^y$  for functions  $y$  and  $\psi$  satisfying the conditions:  $y(0) \in M$ ,  $\psi(0) = 0$ . The goal of this section is to prove the existence of a limit (given below) with respect to the family of bounded continuous cylindrical functions, where by a cylindrical function  $C([0, 1] \times [0, 1], \mathbb{R}^m) \rightarrow \mathbb{R}$  we mean a function  $f$  for which there exists a finite collection of points  $\tau_1, \dots, \tau_n, \xi_1, \dots, \xi_k$  and a function  $\tilde{f}: \mathbb{R}^{nk} \rightarrow \mathbb{R}$  such that

$$f(\omega) = \tilde{f}(\omega(\tau_1, \xi_1), \omega(\tau_1, \xi_2), \dots, \omega(\tau_n, \xi_k)).$$

This limit defines the measure  $\widetilde{\mathbb{W}}_{M,\psi,s,t}^y$ :

$$\int_{C([0,s], \mathbb{R}^m)^t} f(\omega) \widetilde{\mathbb{W}}_{M,\psi,s,t}^y(d\omega) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{E}_{y,\psi} \{ f(\omega) \mathbb{I}_{\{(\mathbf{w}_\psi^y)_t(s) \in U_\varepsilon(M)\}} \}}{\widetilde{\mathbb{W}}_\psi^y \{ (\mathbf{W}_\psi^y)_t(s) \in U_\varepsilon(M) \}}. \quad (1)$$

Before proving the existence of such a limit, we consider the process

$$(\mathbf{W}_{\psi,s}^z)_t = \psi(t) + B_t^s,$$

where  $\psi: [0, 1] \rightarrow \mathbb{R}^m$  is a continuous function satisfying the condition  $\psi(0) = 0$  and  $B_t^s$  is a Brownian motion with parameter  $s$  issuing from the point  $z$ . The results obtained for this process will be used for a subsequent construction.

**Some results for the process  $(\mathbf{W}_{\psi,s}^z)_t$ .** Suppose that  $\mathbb{W}_{\psi,s}^z$  denotes the distribution of the process  $(\mathbf{W}_{\psi,s}^z)_t$  and  $\mathbb{E}_{z,\psi,s}$  is the expectation with respect to this distribution.

**Lemma 1.** *The limit*

$$\int_{C([0,t],\mathbb{R}^m)} f(\omega) \mathbb{W}_{M,\psi,s,t}^z(d\omega) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}_{z,\psi,s} \{ f(\omega) \mathbb{I}_{\{(\mathbf{W}_{\psi,s}^z)_t \in U_\varepsilon(M)\}} \}}{\mathbb{W}_{\psi,s}^z \{ (\mathbf{W}_{\psi,s}^z)_t \in U_\varepsilon(M) \}},$$

with respect to the family of continuous bounded cylindrical functions, exists and defines the measure  $\mathbb{W}_{M,\psi,s,t}^z$  in the integral on the left.

**Sketch of the proof.** Let us find a function  $\tilde{f}: \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  and a finite set of points  $\tau_1, \dots, \tau_k$  such that

$$f(\omega) = \tilde{f}(\omega(\tau_1), \dots, \omega(\tau_k), \omega(t)).$$

We have

$$\begin{aligned} \int_{C([0,t],\mathbb{R}^m)} f(\omega) \mathbb{W}_{M,\psi,s,t}^z(d\omega) &= \lim_{\varepsilon \rightarrow 0} \frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega) \mathbb{I}_{\{\omega : \omega(t) \in U_\varepsilon(M)\}} \mathbb{W}_{\psi,s}^z(d\omega)}{\mathbb{W}_{\psi,s}^z \{ \omega : \omega(t) \in U_\varepsilon(M) \}} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\mathbf{P}^{\mathbb{W}}(t, 0, U_\varepsilon(M - z - \psi(t)))} \int_{\mathbb{R}^m} \mathbf{P}^{\mathbb{W}}(\tau_1, 0, dx_1) \int_{\mathbb{R}^m} \mathbf{P}^{\mathbb{W}}(\tau_2 - \tau_1, x_1, dx_2) \cdots \\ &\quad \times \int_{U_\varepsilon(M - \psi(t) - z)} \mathbf{P}^{\mathbb{W}}(t - \tau_k, x_k, dx_{k+1}) \\ &\quad \times \tilde{f}(x_1 + z + \psi(\tau_1), \dots, x_k + z + \psi(\tau_k), x_{k+1} + z + \psi(t)), \end{aligned}$$

where

$$\mathbf{P}^{\mathbb{W}}(\tau, x, dz) = \frac{1}{(2\pi s\tau)^{m/2}} \exp\left\{-\frac{|z-x|^2}{2s\tau}\right\} dz.$$

Since the function in the integrand is bounded, it suffices, by Lebesgue's theorem, to prove that the following limit exists:

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \frac{\int_{U_\varepsilon(M - \psi(t) - z)} \tilde{f}(x_1 + z + \psi(\tau_1), \dots, x_{k+1} + z + \psi(t)) \mathbf{P}^{\mathbb{W}}(t - \tau_k, x_k, dx_{k+1})}{\mathbf{P}^{\mathbb{W}}(t, 0, U_\varepsilon(M - z - \psi(t)))} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_{U_\varepsilon(M - \psi(t) - z - x_k)} \tilde{f}(x_1 + z + \psi(\tau_1), \dots, x_{k+1} + x_k + z + \psi(t)) \mathbf{P}^{\mathbb{W}}(t - \tau_k, 0, dx_{k+1})}{\mathbf{P}^{\mathbb{W}}(t, 0, U_\varepsilon(M - z - \psi(t)))}. \end{aligned}$$

By  $M_1$  we denote the manifold  $M - \psi(t) - z - x_k$  and by  $M_2$  the manifold  $M - \psi(t) - z$ . Further, suppose that

$$\lambda_\varepsilon = \frac{1}{\text{vol}_{m-d}(\varepsilon)} l|_{U_\varepsilon(M_1)}, \quad \mu_\varepsilon = \frac{1}{\text{vol}_{m-d}(\varepsilon)} l|_{U_\varepsilon(M_2)},$$

where  $l$  is the Lebesgue measure on  $\mathbb{R}^m$ . We can easily see that the proof of the existence of this limit can be reduced to that of the existence of the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{\mathbb{R}^m} g(x_{k+1}) \exp\left\{-\frac{|x_{k+1}-x_k|^2}{2s(t-\tau_k)}\right\} \lambda_\varepsilon(dx_{k+1})}{\int_{\mathbb{R}^m} \exp\left\{-\frac{|x_{k+1}|^2}{2st}\right\} \mu_\varepsilon(dx_{k+1})},$$

where  $g: \mathbb{R} \rightarrow \mathbb{R}$  is another symbol for the function  $\tilde{f}$  introduced to indicate the dependence on the last variable solely. We can easily show that, as  $\varepsilon \rightarrow 0$ , the measures  $\lambda_\varepsilon$  and  $\mu_\varepsilon$  converge weakly to the surface measures on  $M_1$  and  $M_2$ , respectively.  $\square$

**Lemma 2.** *The limit (1) with respect to the family of continuous bounded cylindrical functions exists.*

**Sketch of the proof.** Suppose that  $P^{\tilde{W}}(t, y, \Gamma) = \tilde{W}^y(\omega : \omega(t) \in \Gamma)$  is the transition probability for the measure  $\tilde{W}^y$ , where  $y \in C([0, 1], \mathbb{R}^m)$ . Further, suppose that the function

$$\tilde{f}: C([0, s], \mathbb{R}^m)^{k+1} \rightarrow \mathbb{R}$$

and the finite set of points  $\tau_1, \tau_2, \dots, \tau_k$  satisfy the relation

$$f(\omega) = \tilde{f}(\omega(\tau_1), \omega(\tau_2), \dots, \omega(\tau_k), \omega(t)).$$

Let the symbol  $\pi_s$  denote the coordinate mapping. The proof is carried out by using the following formula from [2, p. 204]:

$$\begin{aligned} \int_{C([0, s], \mathbb{R}^m)^t} f(\omega) \tilde{W}^0(d\omega) &= \int_{C([0, s], \mathbb{R}^m)} P^{\tilde{W}}(\tau_1, 0, dw_1) \int_{C([0, s], \mathbb{R}^m)} P^{\tilde{W}}(\tau_2 - \tau_1, w_1, dw_2) \cdots \\ &\times \int_{\pi_s^{-1}(U_\varepsilon(M - \psi(t) - y(s)))} \tilde{f}(w_1, \dots, w_{k+1}) P^{\tilde{W}}(t - \tau_k, w_k, dw_{k+1}) \end{aligned}$$

and applying Lemma 1 to the measure in the last integral.  $\square$

## 2. ASYMPTOTICS IN $t$ FOR AN INTEGRAL OF SPECIFIC FORM

**Proposition 1.** *Let  $i$  be an isometric embedding of the manifold  $M$  in  $\mathbb{R}^m$  and  $g \in C^2(M)$ . Then*

$$\frac{1}{(2\pi t)^{d/2}} \int_M g(z) \exp\left\{-\frac{|z-y|^2}{2t}\right\} \lambda_M(dz) = g(y) + \frac{t}{8} g(y)(c(y) - \text{scal}(y)) - \frac{t}{2} \Delta_M g(y) + tR(t, y),$$

where  $|R(t, y)| < Kt^{1/2}$ ,  $K$  is a constant independent of  $y$ ,  $\text{scal}(y)$  is the scalar curvature at the point  $y$ , and the function  $c(y)$  is of the form

$$c(y) = \sum_{k,l} \sum_{\alpha} \left( \frac{\partial^2 i^\alpha}{\partial x^k \partial x^l} \right)^2 (0),$$

where the  $x^k$  are the normal coordinates in a neighborhood  $U_y$  of the point  $y$  which are specified by the homeomorphism of the neighborhood  $U_y$  onto a neighborhood of zero  $U$  in  $\mathbb{R}^d$ . Independently of the local coordinates,  $c(y)$  can be written as

$$c(y) = -\frac{1}{2} \Delta_M \Delta_M |y - \cdot|^2|_y - \frac{1}{3} \text{scal}(y)$$

and, therefore,  $c(y)$  depends only on the embedding  $i$ .

**Sketch of the proof.** We have obtained a more exact asymptotic expression in comparison with that obtained for an integral of similar form in [3]. The idea of the proof is the same.  $\square$

**Corollary 1.** *Suppose that  $g \in C^2(M)$ . Then the following asymptotics is valid:*

$$\frac{\int_M g(z) \exp\left\{-\frac{|z-y|^2}{2t}\right\} \lambda_M(dz)}{\int_M \exp\left\{-\frac{|z-y|^2}{2t}\right\} \lambda_M(dz)} = g(y) - \frac{t}{2} \Delta_M g(y) + tR_1(t, y),$$

where  $|R_1(t, y)| < K_1 t^{1/2}$ , and  $K_1$  is a constant independent of  $y$ .

**Corollary 2.** *Suppose that  $g \in C^2(M)$ ,  $y \in M$ , and  $\psi$  is a Hölder function of Hölder order  $\alpha$ ,  $1/3 < \alpha < 1/2$ , such that  $\psi(0) = 0$ . Suppose that  $\text{Pr}_M$  is the projection mapping onto the manifold  $M$  along the subspaces normal to the manifold and defined in a suitable neighborhood of the manifold  $\psi_M(t, y) = \text{Pr}_M(y + \psi(t))$ . Then the following asymptotics is valid:*

$$\frac{\int_M g(z) \exp\left\{-\frac{|z-y-\psi(t)|^2}{2t}\right\} \lambda_M(dz)}{\int_M \exp\left\{-\frac{|z-y-\psi(t)|^2}{2t}\right\} \lambda_M(dz)} = g(y + \psi_M(t)) - \frac{t}{2} \Delta_M g(y) + tR_2(t, y),$$

where  $|R_2(t, y)| < K_2 t^{3\alpha-1}$  and  $K_2$  is a constant.

### 3. SECOND STEP IN THE CONSTRUCTION OF THE RANDOM FIELD $\mathbf{W}_M^x$

Suppose that  $f$  is a continuous bounded cylindrical function on  $C([0, s], \mathbb{R}^m)^1$  and  $\varphi: \mathbb{R} \rightarrow M$  is a function which is the trajectory of the Brownian motion on  $M$  such that  $\varphi(0) = x$ . Suppose that  $\mathcal{P}_1 = \{0 = t_0 \leq t_1 \leq \dots \leq t_n = 1\}$  is a partition of the interval  $[0, 1]$ . If  $E$  is a locally convex space, then to each  $\omega \in E^1$  we can assign a finite sequence of  $n$  elements

$$(\omega_1, \omega_2, \dots, \omega_n) \in E^{t_1} \times E^{t_2-t_1} \times \dots \times E^{t_n-t_{n-1}},$$

where  $\omega_j$  is defined on the interval  $[0, t_j - t_{j-1}]$  by the formula  $\omega_j(t) = \omega(t_{j-1} + t)$ . We define the function  $\varphi_{t_{i-1}t_i}$  on the interval  $[0, t_i - t_{i-1}]$  as follows:

$$\varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1} + t) - \varphi(t_{i-1}).$$

Let us define the measure  $\widetilde{\mathbb{W}}_{M, \varphi, s, \mathcal{P}_1}^x$  by the formula

$$\begin{aligned} \int_{C([0, s], \mathbb{R}^m)^1} f(\omega) \widetilde{\mathbb{W}}_{M, \varphi, s, \mathcal{P}_1}^x(d\omega) &= \int_{C([0, s], \mathbb{R}^m)^{t_1}} \widetilde{\mathbb{W}}_{M, \varphi_{0t_1}, s, t_1}^x(d\omega_1) \\ &\times \int_{C([0, s], \mathbb{R}^m)^{t_2-t_1}} \widetilde{\mathbb{W}}_{M, \varphi_{t_1t_2}, s, t_2-t_1}^{\omega_1(t_1)}(d\omega_2) \dots \\ &\times \int_{C([0, s], \mathbb{R}^m)^{t_n-t_{n-1}}} \widetilde{\mathbb{W}}_{M, \varphi_{t_{n-1}t_n}, s, t_n-t_{n-1}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}(d\omega_n) f(\omega_1, \omega_2, \dots, \omega_n). \end{aligned}$$

It is readily verified that  $\omega_i(t_i - t_{i-1})(0) \in M$ , so that the measure  $\widetilde{\mathbb{W}}_{M, \varphi, s, \mathcal{P}_1}^x$  is well defined. Further, let  $\mathcal{P}_2 = \{0 = s_0 \leq s_1 \leq \dots \leq s_k = 1\}$  be a partition of the interval  $[0, 1]$ . Now, suppose that  $s$  is a time parameter. Instead of the symbol  $\widetilde{\mathbb{W}}_{M, \varphi, s, \mathcal{P}_1}^x$ , we shall write  $\widetilde{\mathbb{W}}_{M, s, \mathcal{P}_1}^\varphi$ . Let us define the measure  $\mathbb{W}_{M, \mathcal{P}_1, \mathcal{P}_2}^x$  by the formula

$$\begin{aligned} \int_{C([0, 1], \mathbb{R}^m)^1} f(\omega) \mathbb{W}_{M, \mathcal{P}_1, \mathcal{P}_2}^x(d\omega) &= \int_{C([0, 1], \mathbb{R}^m)^{s_1}} \widetilde{\mathbb{W}}_{M, s_1, \mathcal{P}_1}^x(d\omega_1) \\ &\times \int_{C([0, 1], \mathbb{R}^m)^{s_2-s_1}} \widetilde{\mathbb{W}}_{M, s_2-s_1, \mathcal{P}_1}^{\omega_1(s_1)}(d\omega_2) \dots \\ &\times \int_{C([0, 1], \mathbb{R}^m)^{s_n-s_{n-1}}} \widetilde{\mathbb{W}}_{M, s_n-s_{n-1}, \mathcal{P}_1}^{\omega_{n-1}(s_{n-1}-s_{n-2})}(d\omega_n) f(\omega_1, \dots, \omega_n). \end{aligned}$$

**Theorem 1.** For each  $x \in M$ , if the meshes of the partitions  $\mathcal{P}_1$  and  $\mathcal{P}_2$  tend to zero, then the sequence of measures  $\mathbb{W}_{M, \mathcal{P}_1, \mathcal{P}_2}^x$  is weakly convergent to the measure  $\mathbb{W}_M^x$  with respect to the family of continuous bounded cylindrical functions. The measure  $\mathbb{W}_M^x$  regarded as the distribution of a process with values in  $C([0, 1], M)$ , possesses a transition probability at time  $t$  coinciding with the distribution of the Brownian motion with parameter  $t$  on the manifold issuing from the point  $x$ .

**Sketch of the proof.** Suppose that

$$\begin{aligned} h \cdot \mathbb{W}_{M, \varphi, s, \mathcal{P}_1}^x & \left( = h \cdot \mathbb{W}_{M, s, \mathcal{P}_1}^\varphi \right) = \int_{C([0, 1], \mathbb{R}^m)} h(\omega) \mathbb{W}_{M, \varphi, s, \mathcal{P}_1}^x(d\omega) \\ & = \int_{C([0, t_1], \mathbb{R}^m)} \mathbb{W}_{M, \varphi_{0t_1}, s, t_1}^x(dw_1) \int_{C([0, t_2-t_1], \mathbb{R}^m)} \mathbb{W}_{M, \varphi_{t_1 t_2}, s, t_2-t_1}^{\omega_1(t_1)}(dw_2) \cdots \\ & \quad \times \int_{C([0, t_n-t_{n-1}], \mathbb{R}^m)} \mathbb{W}_{M, \varphi_{t_{n-1} t_n}, s, t_n-t_{n-1}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}(dw_n) h(\omega_1, \omega_2, \dots, \omega_n). \end{aligned}$$

Suppose that there exists a function  $\tilde{f}: C([0, 1], \mathbb{R}^m) \rightarrow \mathbb{R}$  such that  $f(\omega) = \tilde{f}(\omega(t))$ . Then

$$\begin{aligned} \int_{C([0, s], \mathbb{R}^m)^1} f(\omega) \widetilde{\mathbb{W}}_{M, s, \mathcal{P}_1}^\varphi(d\omega) & = \int_{C([0, 1], \mathbb{R}^m)} \tilde{f}(w) \widetilde{\mathbb{W}}_{M, s, \mathcal{P}_1}^\varphi \circ \pi_s^{-1}(dw) \\ & = \int_{C([0, 1], \mathbb{R}^m)} \tilde{f}(w) \mathbb{W}_{M, s, \mathcal{P}_1}^\varphi(dw). \end{aligned}$$

This yields

$$\begin{aligned} \int_{C([0, 1], \mathbb{R}^m)^1} f(\omega) \mathbb{W}_{M, \mathcal{P}_1, \mathcal{P}_2}^x(d\omega) & = \int_{C([0, 1], \mathbb{R}^m)} \mathbb{W}_{M, s_1, \mathcal{P}_1}^x(dw_1) \\ & \quad \times \int_{C([0, 1], \mathbb{R}^m)} \mathbb{W}_{M, s_2-s_1, \mathcal{P}_1}^{w_1}(dw_2) \cdots \int_{C([0, 1], \mathbb{R}^m)} \mathbb{W}_{M, s_n-s_{n-1}, \mathcal{P}_1}^{w_{n-1}}(dw_{n-1}) \\ & \quad \times \int_{C([0, 1], \mathbb{R}^m)} \mathbb{W}_{M, s_n-s_{n-1}, \mathcal{P}_1}^{w_{n-1}}(dw_n) \tilde{f}(w_n). \end{aligned}$$

Consider the integral

$$\int_{C([0, t], \mathbb{R}^m)} g(\omega) \mathbb{W}_{M, \psi, s, t}^z(d\omega),$$

where the function  $g \in C([0, t], \mathbb{R}^m)$  is such that there exists a function  $\tilde{g} \in C(\mathbb{R})$  for which  $g(\omega) = \tilde{g}(\omega(t))$ . As a result of simple calculations, we obtain

$$\begin{aligned} \int_{C([0, t], \mathbb{R}^m)} g(\omega) \mathbb{W}_{M, \psi, s, t}^z(d\omega) & = \lim_{\varepsilon \rightarrow 0} \frac{\int_{C([0, t], \mathbb{R}^m)} g(\omega) \mathbb{I}_{\{\omega : \omega(t) \in U_\varepsilon(M)\}}(\omega) \mathbb{W}_{\psi, s}^z(d\omega)}{\mathbb{W}_{\psi, s}^z\{\omega : \omega(t) \in U_\varepsilon(M)\}} \\ & = \frac{\int_M \exp\left\{-\frac{|x_1 - z - \psi(t)|^2}{2ts}\right\} \tilde{g}(x_1) \lambda_M(dx_1)}{\int_M \exp\left\{-\frac{|x_1 - z - \psi(t)|^2}{2ts}\right\} \lambda_M(dx_1)}. \end{aligned}$$

First, suppose that the function  $f$  is such that there exists a function  $p: \mathbb{R}^m \rightarrow \mathbb{R}$  and numbers  $t, s \in [0, 1]$  for which  $f(\omega) = p(\omega(t, s))$ . The integral

$$\int_{C([0, 1], \mathbb{R}^m)^1} f(\omega) \mathbb{W}_{M, \mathcal{P}_1, \mathcal{P}_2}^x(d\omega)$$

is of the form

$$\begin{aligned} & \frac{\int_M \exp\left\{-\frac{|x_1-x|^2}{2\Delta s_1 \Delta t_1}\right\} dx_1}{\int_M \exp\left\{-\frac{|\bar{x}_1-x|^2}{2\Delta s_1 \Delta t_1}\right\} d\bar{x}_1} \cdots \frac{\int_M \exp\left\{-\frac{|x_{n-1}-x_{n-2}|^2}{2\Delta s_1 \Delta t_{n-1}}\right\} dx_{n-1}}{\int_M \exp\left\{-\frac{|\bar{x}_{n-1}-x_{n-2}|^2}{2\Delta s_1 \Delta t_{n-1}}\right\} d\bar{x}_{n-1}} \frac{\int_M \exp\left\{-\frac{|x_n-x_{n-1}|^2}{2\Delta s_1 \Delta t_n}\right\} dx_n}{\int_M \exp\left\{-\frac{|\bar{x}_n-x_{n-1}|^2}{2\Delta s_1 \Delta t_n}\right\} d\bar{x}_n} \\ & \times \frac{\int_M \exp\left\{-\frac{|y_1-x_1|^2}{2\Delta s_2 \Delta t_1}\right\} dy_1}{\int_M \exp\left\{-\frac{|\bar{y}_1-x_1|^2}{2\Delta s_2 \Delta t_1}\right\} d\bar{y}_1} \cdots \frac{\int_M \exp\left\{-\frac{|y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2 \Delta t_{n-1}}\right\} dy_{n-1}}{\int_M \exp\left\{-\frac{|\bar{y}_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2 \Delta t_{n-1}}\right\} d\bar{y}_{n-1}} \\ & \times \frac{\int_M \exp\left\{-\frac{|y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2 \Delta t_n}\right\} dy_n}{\int_M \exp\left\{-\frac{|\bar{y}_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2 \Delta t_n}\right\} d\bar{y}_n} \cdots \\ & \times \frac{\int_M \exp\left\{-\frac{|u_1-z_1|^2}{2\Delta s_{k-1} \Delta t_1}\right\} du_1}{\int_M \exp\left\{-\frac{|\bar{u}_1-z_1|^2}{2\Delta s_{k-1} \Delta t_1}\right\} d\bar{u}_1} \cdots \frac{\int_M \exp\left\{-\frac{|u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1} \Delta t_{n-1}}\right\} du_{n-1}}{\int_M \exp\left\{-\frac{|\bar{u}_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1} \Delta t_{n-1}}\right\} d\bar{u}_{n-1}} \\ & \times \frac{\int_M \exp\left\{-\frac{|u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1} \Delta t_n}\right\} du_n}{\int_M \exp\left\{-\frac{|\bar{u}_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1} \Delta t_n}\right\} d\bar{u}_n} \\ & \times \frac{\int_M \exp\left\{-\frac{|v_1-u_1|^2}{2\Delta s_k \Delta t_1}\right\} dv_1}{\int_M \exp\left\{-\frac{|\bar{v}_1-u_1|^2}{2\Delta s_k \Delta t_1}\right\} d\bar{v}_1} \cdots \frac{\int_M \exp\left\{-\frac{|v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k \Delta t_{n-1}}\right\} dv_{n-1}}{\int_M \exp\left\{-\frac{|\bar{v}_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k \Delta t_{n-1}}\right\} d\bar{v}_{n-1}} \\ & \times \frac{\int_M \exp\left\{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k \Delta t_n}\right\} p(v_n) dv_n}{\int_M \exp\left\{-\frac{|\bar{v}_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k \Delta t_n}\right\} d\bar{v}_n}, \end{aligned}$$

where  $\Delta t_i = t_i - t_{i-1}$ ,  $\Delta s_j = s_j - s_{j-1}$ , and, to simplify the notation, instead of  $\lambda_M(dz)$  we use  $dz$ . We have also assumed that  $t_n = t$  and  $s_k = s$ . Let us denote this integral by  $I(\mathcal{P}_1, \mathcal{P}_2, p)$ .

**Lemma 3.** *The integral  $I(\mathcal{P}_1, \mathcal{P}_2, p)$  converges to  $\exp\{-\frac{st}{2} \Delta_M\} p$  if the meshes of  $|\mathcal{P}_1|$  and  $|\mathcal{P}_2|$  tend to zero.*

**Sketch of the proof.** Using Corollaries 1 and 2 of Proposition 1, we obtain the following asymptotics for the integral  $I(\mathcal{P}_1, \mathcal{P}_2, p)$ :

$$I(\mathcal{P}_1, \mathcal{P}_2, p)(x) = p(x) - \frac{st}{2} \Delta_M p(x) + O(s^2 t^2) + O(|\mathcal{P}_2|, |\mathcal{P}_1|).$$

Hence we see that the following limit exists:

$$\lim_{|\mathcal{P}_1| \rightarrow 0, |\mathcal{P}_2| \rightarrow 0} I(\mathcal{P}_1, \mathcal{P}_2, p)(x) = p(x) - \frac{st}{2} \Delta_M p(x) + O(s^2 t^2) (= (Q_{stp})(x)).$$

Further, we verify that  $Q_{\tau+\Delta\tau} = Q_\tau Q_{\Delta\tau}$ . Let  $s$  and  $t$  satisfy  $\tau = st$ . Let us find a  $\Delta s$  and  $\Delta t$  such that  $\tau + \Delta\tau = (s + \Delta s)(t + \Delta t)$ , and consider the integral  $I(\mathcal{P}_1^{[0, t+\Delta t]}, \mathcal{P}_2^{[0, s+\Delta s]}, p)$  for partitions of the closed intervals  $[0, s + \Delta s]$  and  $[0, t + \Delta t]$  resulting from supplementing the corresponding partitions of the closed intervals  $[0, s]$  and  $[0, t]$  by points of partitions of the closed intervals  $[t, t + \Delta t]$ ,  $[s, s + \Delta s]$ . We can easily see from the structure of the integral  $I(\mathcal{P}_1, \mathcal{P}_2, p)$  that

$$I(\mathcal{P}_1^{[0, t+\Delta t]}, \mathcal{P}_2^{[0, s+\Delta s]}, p) = I(\mathcal{P}_1^{[0, t]}, \mathcal{P}_2^{[0, s]}, I(\mathcal{P}_1^{[t, t+\Delta t]}, \mathcal{P}_2^{[s, s+\Delta s]}, p)).$$

Passing to the limit in this expression and taking  $\Delta\tau = s\Delta t + t\Delta s + \Delta s\Delta t$  into account, we find that  $Q_\tau$  is a semigroup which, by the proof above, satisfies

$$(Q_\tau p)(x) = p(x) - \frac{\tau}{2} \Delta_M p(x) + O(\tau^2).$$

Thus, we find that

$$Q_\tau = \exp\left\{-\frac{\tau}{2} \Delta_M\right\}. \quad \square$$

For a function  $f$  depending on  $\omega$  at several points, such as at points  $\xi_i \in [0, s]$  and  $\tau_j \in [0, t]$ , the integral  $I(\mathcal{P}_1, \mathcal{P}_2, p)$  will be of the same form. The convergence examined above occurs on each square  $[\xi_{i-1}, \xi_i] \times [\tau_{j-1}, \tau_j]$ . Each of operators of the form

$$\exp\left\{-\frac{\Delta\xi_i \Delta\tau_j}{2} \Delta_M\right\}$$

acts on the corresponding variable of the function  $p$  defined as

$$f(\omega) = p(\omega_{11}(\xi_1, \tau_1), \dots, \omega_{kl}(\xi_l, \tau_l)),$$

where  $\omega_{ij}$  is defined on  $[0, \xi_i - \xi_{i-1}] \times [0, \tau_j - \tau_{j-1}]$  by the formula

$$\omega_{ij}(s, t) = \omega(\xi_{i-1} + s, \tau_{j-1} + t). \quad \square$$

**Corollary 3.** *Suppose that  $M$  is a compact Lie group. Then the random field  $\mathbf{W}_M^x$  regarded as a process with values in  $C([0, 1], M)$ , coincides with the Brownian motion constructed in [1].*

**Sketch of the proof.** The proof follows from Theorem 1 and Theorem 2.15 from [4] (Theorem 2.15 from [4] was also proved in Lemma 3.3 from [5]).  $\square$

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M. V. LOMONOSOV MOSCOW STATE UNIVERSITY