Constructing a Brownian sheet on a compact Riemannian manifold

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Outline

- Formulation of results
- Approach
- Three steps of the construction of the measure $\mathbb{W}_M^x$
- Outline of proof
- Improved result of P. Malliavin on the construction of a Brownian sheet on a compact Lie group (1991)
- Conclusion and open problems

Formulation of results

Let M be a compact Riemannian manifold without boundary isometrically embedded into $\mathbb{R}^m$.

Let $\mathcal{P}_1 \times \mathcal{P}_2$ be a partition of the square $[0,1] \times [0,1]$, where $\mathcal{P}_1 = \{0 = t_0 < \cdots < t_n = 1\}$, $\mathcal{P}_2 = \{0 = s_0 < \cdots < s_k = 1\}$

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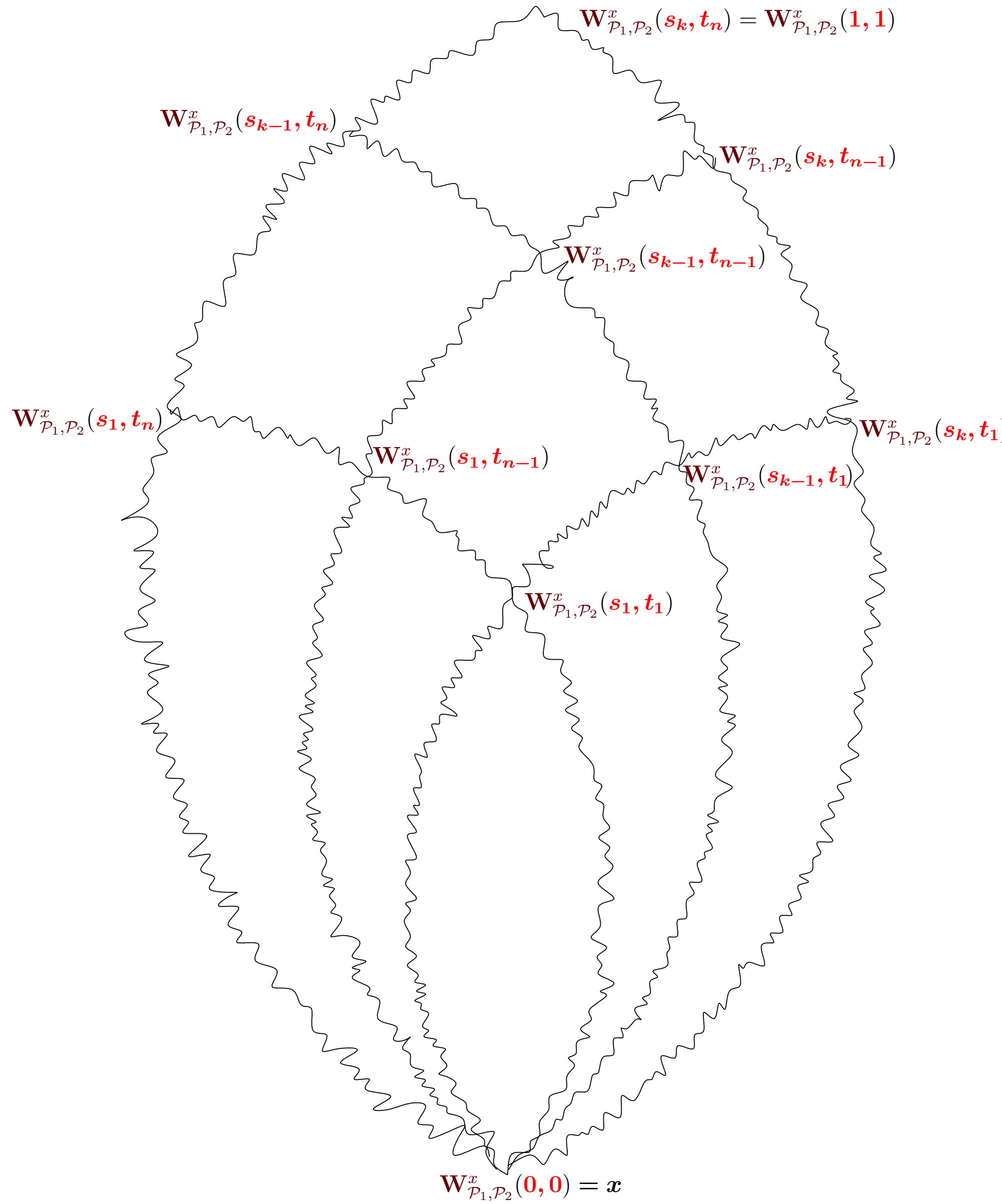
Let $\mathbf{W}^x_{\mathcal{P}_1,\mathcal{P}_2}(s,t)$ denote a process starting at some point $x \in M$, behaving on each square $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ as a Brownian sheet on $\mathbb{R}^m$ conditioned to return to M at all points $(s_i, t_j) \in \mathcal{P}_1 \times \mathcal{P}_2$.

Brownian sheet — picture for an intuitive understanding



Formulation of results

THEOREM

For every $x \in M$, as $|\mathcal{P}_1| \longrightarrow 0$ and $|\mathcal{P}_2| \longrightarrow 0$, the distributions of $W^x_{\mathcal{P}_1,\mathcal{P}_2}$ converge weakly, relative to the family of bounded continuous cylinder functions, to a measure $\mathbb{W}^x_M$. The measure $\mathbb{W}^x_M$ has the following properties:

(i) *considered as the distribution of a $\mathrm{C}([0,1], M)$-valued process, $\mathbb{W}^x_M$ possesses a transition probability at time t which is the distribution of a Brownian motion on M with variance t starting at the point x;*

(ii) *considered as the distribution of an M-valued two-parameter stochastic process, $\mathbb{W}^x_M$ possesses a transition probability at time (s, t) which is the heat kernel measure on M at time st, i.e., if $f \in \mathrm{C}(M)$, then*

$$\mathbb{W}^x_M \circ \pi^{-1}_{s,t}[f] = \left(e^{-\frac{st}{2}\Delta_M} f\right)(x),$$

where $\pi_{s,t}$ is the evaluation mapping $\mathrm{C}([0,1] \times [0,1], M) \longrightarrow M, \omega \mapsto \omega(s,t)$, Δ_M is the Laplace–Beltrami operator on M, and $e^{-\frac{\tau}{2}\Delta_M}$ is the heat semigroup.

Formulation of results

COROLLARY

Let M be a compact Lie group. Then, the stochastic field $\mathbf{W}_M^x$ with distribution $\mathbb{W}_M^x$, considered as a $\mathrm{C}([0,1], M)$-valued process, coincides with the Brownian motion on M constructed by P. Malliavin [a]

[a] Malliavin, P., Hypoellipticity in infinite dimensions, Diffusion processes and related problems in analysis., Vol.1 (Evanston, IL, 1989), Birkhäuser Boston, Boston, MA, 1990, pp. 17-31

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- The method of construction of the stochastic field $\mathbf{W}_M^x$ is very different to the method of Malliavin

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O. G. Smolyanov, H. v. Weizsäcker, and their coauthors (O. Wittich, N. Sidorova) investigated the approximation of a Brownian motion with values in a compact Riemannian manifold M by conditioned Brownian motions on the ambient Euclidean space $\mathbb{R}^m$.

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Their approach consists of the following stages:

- consideration of the infinite dimensional manifold $\mathrm{C}([0,1], M)$;
- the manifold $\mathrm{C}([0,1], M)$ is considered to be embedded into $\mathrm{C}([0,1], \mathbb{R}^m)$;
- consideration of the distribution of an M-valued Brownian motion as a surface measure on $\mathrm{C}([0,1], M)$;
- approximation of this surface measure by “volume” measures on $\mathrm{C}([0,1], \mathbb{R}^m)$, where “volume” measures on $\mathrm{C}([0,1], \mathbb{R}^m)$ are distributions of $\mathbb{R}^m$-valued conditioned Brownian motions.

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- we consider the distribution of an M-valued Brownian sheet as a surface measure on $\mathrm{C}([0,1] \times [0,1], M)$

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Namely

- we consider the distribution of an M-valued Brownian sheet as a surface measure on $\mathrm{C}([0,1] \times [0,1], M)$
- we approximate this surface measure by distributions of stepwise conditioned Gaussian stochastic fields on the Euclidean space $\mathbb{R}^m$ enveloping the manifold M

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Let
$\mathbf{W}_{t,s}$ be a Brownian sheet on $\mathbb{R}^m$,
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$\mathbb{W}^y$ denote the distribution of the process $\mathbf{W}_t^y = y + \mathbf{W}_t$

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For $\psi \in \mathrm{C}([0,1],\mathbb{R}^m)$, define
$(\mathbf{W}^y_\psi)_t = \psi(t) + \mathbf{W}^y_t$

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Let
$\tilde{\mathbb{W}}_\psi^y$ be the distribution of $(\mathbf{W}_\psi^y)_t$

$\mathbb{E}_{y,\psi}$ be the expectation relative to the measure $\tilde{\mathbb{W}}_\psi^y$

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Let M be a d-dimensional compact Riemannian manifold without boundary isometrically embedded into $\mathbb{R}^m$

$U_\varepsilon(M)$ be the ε-neighborhood of M

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By a cylinder function f on $\mathrm{C}([0,1]\times[0,1],\mathbb{R}^m) \to \mathbb{R}$, we understand a function for which one can find a finite number of points $\tau_1,\ldots,\tau_n$, $\xi_1,\ldots,\xi_k$ and a function $\tilde{f} : \mathbb{R}^{nk} \to \mathbb{R}$ such that

$$f(\omega) = \tilde{f}(\omega(\tau_1,\xi_1),\omega(\tau_1,\xi_2),\ldots,\omega(\tau_n,\xi_k))$$

First Step of Construction of $\mathbb{W}_M^x$

Measure $\tilde{\mathbb{W}}^y_{\psi,s,t}$: $y(0) \in M, \psi(0) = 0, (s,t) \in [0,1] \times [0,1]$

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$$\int_{\mathrm{C}([0,s],\mathbb{R}^m)^t} f(\omega) \tilde{\mathbb{W}}^y_{\psi,s,t}(d\omega) = \lim_{\varepsilon \to 0} \frac{\mathbb{E}_{y,\psi}\{f(\omega)\, \mathbb{I}_{\{(\mathbf{W}^y_\psi)_t(s) \in U_\varepsilon(M)\}}\}}{\tilde{\mathbb{W}}^y_\psi\{(\mathbf{W}^y_\psi)_t(s) \in U_\varepsilon(M)\}}$$

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We can imagine the stochastic field with distribution $\tilde{\mathbb{W}}^y_{\psi,s,t}$ as a Brownian sheet on $\mathbb{R}^m$ conditioned it to return to the manifold at the point (s,t)

Second Step of Construction of $\mathbb{W}_M^x$

Measure $\tilde{\mathbb{W}}^x_{\varphi,s,\mathcal{P}_1}$:

Second Step of Construction of $\mathbb{W}^x_M$

Measure $\tilde{\mathbb{W}}^x_{\varphi,s,\mathcal{P}_1}$:

$\varphi : \mathbb{R} \to M$ Brownian path on M, such that $\varphi(0) = x$

$\mathcal{P}_1 = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ partition of the interval $[0, 1]$

$E^1 \ni \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in E^{t_1} \times E^{t_2 - t_1} \times \cdots \times E^{t_n - t_{n-1}}$

On $[0, t_j - t_{j-1}]$, $\omega_j(t) = \omega(t_{j-1} + t)$, $\quad \varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1} + t) - \varphi(t_{i-1})$

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$$\int\limits_{C([0,s],\mathbb{R}^m)^1} f(\omega)\tilde{\mathbb{W}}^x_{\varphi,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,s],\mathbb{R}^m)^{t_1}} \tilde{\mathbb{W}}^x_{\varphi_{0t_1},s,t_1}(d\omega_1) \int\limits_{C([0,s],\mathbb{R}^m)^{t_2-t_1}} \tilde{\mathbb{W}}^{\omega_1(t_1)}_{\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2)$$

$$\ldots \int\limits_{C([0,s],\mathbb{R}^m)^{t_n-t_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n)\, f(\omega_1, \omega_2, \ldots, \omega_n)$$

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$$\dots \int_{C([0,s],\mathbb{R}^m)^{t_n-t_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n)\, f(\omega_1,\omega_2,\dots,\omega_n)$$

We can imagine the stochastic field with distribution $\tilde{\mathbb{W}}^x_{\varphi,s,\mathcal{P}_1}$ as a Brownian sheet on $\mathbb{R}^m$ which is conditioned to return to the manifold at the points $(s,t_1),(s,t_2),\dots,(s,t_n)$

Third Step of Construction of $\mathbb{W}_M^x$

Measure $\mathbb{W}^x_{\mathcal{P}_1,\mathcal{P}_2}$:

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We can think of the stochastic field with distribution $\mathbb{W}^x_{\mathcal{P}_1,\mathcal{P}_2}$ as of a Brownian sheet on $\mathbb{R}^m$ which is conditioned to return to the manifold at the points of the partition $\mathcal{P}_1 \times \mathcal{P}_2$ of the square $[0,1]\times[0,1]$

Theorem about a Brownian Sheet

THEOREM

For every $x \in M$, as $|\mathcal{P}_1| \longrightarrow 0$ and $|\mathcal{P}_2| \longrightarrow 0$, the distributions of $W^x_{\mathcal{P}_1,\mathcal{P}_2}$ converge weakly, relative to the family of bounded continuous cylinder functions, to a measure $\mathbb{W}^x_M$. The measure $\mathbb{W}^x_M$ has the following properties:

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$f \;:\quad \exists\, p : \mathbb{R}^m \longrightarrow \mathbb{R},\;\; t, s \in [0,1],\; f(\omega) = p(\omega(t,s))$

$$\Longrightarrow \qquad \int\limits_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{\mathcal{P}_1,\mathcal{P}_2}(d\omega) =$$

Theorem about a Brownian Sheet, Outline of Proof

$$\frac{\int\limits_M e^{-\frac{|x_1-x|^2}{2\Delta s_1 \Delta t_1}}\, dx_1}{\int\limits_M e^{-\frac{|\bar{x}_1-x|^2}{2\Delta s_1 \Delta t_1}}\, d\bar{x}_1} \cdots \frac{\int\limits_M e^{-\frac{|x_{n-1}-x_{n-2}|^2}{2\Delta s_1 \Delta t_{n-1}}}\, dx_{n-1}}{\int\limits_M e^{-\frac{|\bar{x}_{n-1}-x_{n-2}|^2}{2\Delta s_1 \Delta t_{n-1}}}\, d\bar{x}_{n-1}} \frac{\int\limits_M e^{-\frac{|x_n-x_{n-1}|^2}{2\Delta s_1 \Delta t_n}}\, dx_n}{\int\limits_M e^{-\frac{|\bar{x}_n-x_{n-1}|^2}{2\Delta s_1 \Delta t_n}}\, d\bar{x}_n}$$

$$\frac{\int\limits_M e^{-\frac{|y_1-x_1|^2}{2\Delta s_2 \Delta t_1}}\, dy_1}{\int\limits_M e^{-\frac{|\bar{y}_1-x_1|^2}{2\Delta s_2 \Delta t_1}}\, d\bar{y}_1} \cdots \frac{\int\limits_M e^{-\frac{|y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2 \Delta t_{n-1}}}\, dy_{n-1}}{\int\limits_M e^{-\frac{|\bar{y}_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2 \Delta t_{n-1}}}\, d\bar{y}_{n-1}} \frac{\int\limits_M e^{-\frac{|y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2 \Delta t_n}}\, dy_n}{\int\limits_M e^{-\frac{|\bar{y}_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2 \Delta t_n}}\, d\bar{y}_n}$$

$$\cdots$$

$$\frac{\int\limits_M e^{-\frac{|u_1-z_1|^2}{2\Delta s_{k-1} \Delta t_1}}\, du_1}{\int\limits_M e^{-\frac{|\bar{u}_1-z_1|^2}{2\Delta s_{k-1} \Delta t_1}}\, d\bar{u}_1} \cdots \frac{\int\limits_M e^{-\frac{|u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1} \Delta t_{n-1}}}\, du_{n-1}}{\int\limits_M e^{-\frac{|\bar{u}_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1} \Delta t_{n-1}}}\, d\bar{u}_{n-1}} \frac{\int\limits_M e^{-\frac{|u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1} \Delta t_n}}\, du_n}{\int\limits_M e^{-\frac{|\bar{u}_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1} \Delta t_n}}\, d\bar{u}_n}$$

$$\frac{\int\limits_M e^{-\frac{|v_1-u_1|^2}{2\Delta s_k \Delta t_1}}\, dv_1}{\int\limits_M e^{-\frac{|\bar{v}_1-u_1|^2}{2\Delta s_k \Delta t_1}}\, d\bar{v}_1} \cdots \frac{\int\limits_M e^{-\frac{|v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k \Delta t_{n-1}}}\, dv_{n-1}}{\int\limits_M e^{-\frac{|\bar{v}_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k \Delta t_{n-1}}}\, d\bar{v}_{n-1}} \frac{\int\limits_M e^{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k \Delta t_n}}\, p(v_n) dv_n}{\int\limits_M e^{-\frac{|\bar{v}_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k \Delta t_n}}\, d\bar{v}_n}$$

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$f \quad : \qquad \exists\, p : \mathbb{R}^m \longrightarrow \mathbb{R},\;\; t, s \in [0,1],\; f(\omega) = p(\omega(t,s))$

$$\int\limits_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{\mathcal{P}_1,\mathcal{P}_2}(d\omega) = I(x, \mathcal{P}_1, \mathcal{P}_2, p)$$

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LEMMA

As $|\mathcal{P}_1| \to 0$ and $|\mathcal{P}_2| \to 0$, $\quad I(x, \mathcal{P}_1, \mathcal{P}_2, p)$ converges to $\left(e^{-\frac{st}{2}\Delta_M} p\right)(x)$.

Theorem about a Brownian Sheet, Outline of Proof

Asymptotics for integrals of Gaussian type

Theorem about a Brownian Sheet, Outline of Proof

Asymptotics for integrals of Gaussian type

PROPOSITION 1

Let $g \in C^2(M)$. Then,

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz) = g(y) - \frac{t}{6}\, g(y)\ \mathrm{scal}(y) - \frac{t}{16}\, g(y)\, \Delta_M \Delta_M\ |y - \cdot\,|^2\Big|_y - \frac{t}{2} \Delta_M g(y) + tR(t,y),$$

where $|R(t,y)| < Kt^{1/2}$, K is a constant, $\mathrm{scal}(y)$ is the scalar curvature at the point y.

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PROPOSITION 2

Let $g \in C^2(M)$, $y \in M$, $0 < t < t_1 < 1$, u_1 and u_2 be such that $|u_2 - u_1| < t_1^\alpha$, $\alpha > 0$. Further, let Pr_M be the mapping of projection onto the manifold M along the subspaces normal to M which are defined in a proper neighborhood of M, $(u_2 - u_1)_M^y$ and $(u_2 - u_1)_\perp^y$ be defined as

$$(u_2 - u_1)_M^y = \mathrm{Pr}_M(y + u_2 - u_1) - y, \quad (u_2 - u_1)_\perp^y = y + u_2 - u_1 - \mathrm{Pr}_M(y + u_2 - u_1).$$

Then, the following asymptotic holds:

$$\frac{\int_M g(z) e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)} = g(y + (u_2 - u_1)_M^y) - \frac{t}{2} \Delta_M g(y + (u_2 - u_1)_M^y)$$
$$+ g(y + (u_2 - u_1)_M^y) \mathcal{R}(y, u_2 - u_1) + t R_2(t, t_1, y, u_2 - u_1);$$

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$$+ g(y+(u_2-u_1)^y_M)\mathcal{R}(y,u_2-u_1) + tR_2(t,t_1,y,u_2-u_1);$$

for the rest terms, we have: $|R_2(t,t_1,y,u_2-u_1)| < K_2 t_1^\alpha$ (K_2 is a constant),

$$\mathcal{R}(y,u_2-u_1) = -\frac{1}{4}(\Delta_M \imath(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)^2$$
$$+\sum_{n=3}^{N}\sum k(n)\prod_{l_1+\cdots+l_s=n} D_M^{2,l_i}(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}\left(y+(u_2-u_1)^y_M\right),$$

where $\imath$ is the isometric embedding of the manifold M into $\mathbb{R}^m$, D_M^{2,l_i} are differential operators on M of the form $D_M^{2,l_i} = (\bigwedge_{k=1}^{l_i}\nabla_M^{(i_k)}, \bigwedge_{k=1}^{l_i}\nabla_M^{(j_k)})$; the number N is chosen so that $t_1^{(N-1)\alpha} < t$.

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- With the help of Proposition 1 and Proposition 2 we get an asymptotic expansion for the integral $I(x, \mathcal{P}_1, \mathcal{P}_2, p)$.

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$$e^{-\frac{st}{2}\Delta_M} = e^{-\frac{\Delta s_1 \Delta t_1}{2}\Delta_M} \cdots e^{-\frac{\Delta s_k \Delta t_n}{2}\Delta_M}$$

applied to p at the point x, where for each of the exponents we use the asymptotic

$$e^{-\frac{\Delta s_i \Delta t_j}{2}\Delta_M} = 1 - \frac{\Delta s_i \Delta t_j}{2}\Delta_M + O((\Delta s_i \Delta t_j)^2)$$

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Let M be a compact Lie group. Then, $\mathbb{W}^x_M$, considered as the distribution of a $\mathrm{C}([0,1], M)$-valued process, coincides with the distribution of a Brownian motion constructed by P. Malliavin

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- $\Longrightarrow$ Coincidence of finite-dimensional distributions
- $\Longrightarrow$ $\mathbb{W}^x_M$ is σ-additive, and coincides with the distribution of a Brownian motion of P. Malliavin

Conclusion

We constructed an M-valued stochastic field $\mathbf{W}_M^x(s,t)$ with distribution $\mathbb{W}_M^x$ such that the stochastic processes $\mathbf{W}_M^x(\,\cdot\,, t)$ and $\mathbf{W}_M^x(s,\,\cdot\,)$ are Brownian motions on M with variances t and s, respectively.

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$\mathbb{W}_M^x$ has been defined only on cylindrical subsets of $\mathrm{C}([0,1],\mathrm{C}([0,1],M))$

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- Heat equation on the manifold $\mathrm{C}([0,1],M)$ is a step towards treating infinite-dimensional Schrödinger equations on manifolds.

Thank you
for your attention