Constructing a Brownian sheet on a compact Riemannian manifold

Abstract

Let M be a compact Riemannian manifold without boundary, isometrically embedded into $\mathbb{R}^m$, and let $\mathcal{P}_1 \times \mathcal{P}_2$ be a partition of $[0,1]\times[0,1]$, where $\mathcal{P}_1 = \{0 = t_0 < \cdots < t_n = 1\}$, $\mathcal{P}_2 = \{0 = s_0 < \cdots < s_k = 1\}$. Let $\mathbf{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(s,t)$ denote a process starting at some point $x \in M$, behaving on each square $[s_{i-1}, s_i]\times[t_{j-1}, t_j]$ as a Brownian sheet on $\mathbb{R}^m$ conditioned to return to M at all points $(s_i, t_j) \in \mathcal{P}_1 \times \mathcal{P}_2$. We obtain the following result:

For every $x \in M$, as $|\mathcal{P}_1| \to 0$ and $|\mathcal{P}_2| \to 0$, the distributions of $\mathbf{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ converge weakly relative to the family of bounded continuous cylinder functions to a measure $\mathbb{W}^x_M$. The measure $\mathbb{W}^x_M$ has the following properties:

(i) *considered as the distribution of a $\mathrm{C}([0,1], M)$-valued process, $\mathbb{W}^x_M$ possesses a transition probability at time t which is the distribution of a Brownian motion on M with variance t starting at the point x;*

(ii) *considered as the distribution of an M-valued two-parameter stochastic process, $\mathbb{W}^x_M$ possesses a transition probability at time (s,t) which is the heat kernel measure on M at time st, i.e., if $f \in \mathrm{C}(M)$, then*

$$\mathbb{W}^x_M \circ \pi^{-1}_{s,t}[f] = \big(e^{-\frac{st}{2}\Delta_M} f\big)(x),$$

where $\pi_{s,t}$ is the mapping $\mathrm{C}([0,1]\times[0,1], M) \to M$, $\omega \mapsto \omega(s,t)$, Δ_M is the Laplace–Beltrami operator on M, $e^{-\frac{\tau}{2}\Delta_M}$ is the heat semigroup.

A corollary of this result generalizes a result of Malliavin [7].

Let M be a compact Lie group. Then, the stochastic field $\mathbf{W}^x_M$ with distribution $\mathbb{W}^x_M$, considered as a $\mathrm{C}([0,1], M)$-valued process, coincides with the Brownian motion on M constructed in [7].

1 The first step of the construction of $\mathbf{W}^x_M$

Let $\mathbf{W}_{t,s}$ be an n-dimensional Brownian sheet. Consider $\mathbf{W}_{t,s}$ as a Brownian motion taking values in the space $C([0,1], \mathbb{R}^m)$ (in the sense of the paper [7])[1]. We denote this process by the symbol $\mathbf{W}_t$. Introduce the following notations: if E is a locally convex space (LCS) then E^t denote $C([0,t], E)$; if $y \in C([0,1], \mathbb{R}^m)$ is a function of a variable s, then $\mathbb{W}^y$ denotes the distribution of the process $\mathbf{W}^y_t = y + \mathbf{W}_t$. If $\psi \in C([0,1], \mathbb{R}^m)$, then for each $t \in [0,1]$ define $(\mathbf{W}^y_\psi)_t = \psi(t) + \mathbf{W}^y_t$. Let $\tilde{\mathbb{W}}^y_\psi$ be the distribution of this process, $\mathbb{E}_{y,\psi}$ be the expectation relative to the measure $\tilde{\mathbb{W}}^y_\psi$. Further, $U_\varepsilon(M)$ denotes the ε-neighborhood of the manifold M. We will consider $\mathbf{W}^y_\psi$ for functions y and ψ satisfying the conditions: $y(0) \in M, \psi(0) = 0$

$$\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\tilde{\mathbb{W}}^y_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{y,\psi}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^y_\psi)_t(s)\in U_\varepsilon(M)\}}\}}{\tilde{\mathbb{W}}^y_\psi\{(\mathbf{W}^y_\psi)_t(s) \in U_\varepsilon(M)\}}\,. \tag{1}$$

[1] A Brownian motion with values in $C([0,1], G)$ where G is a compact Lie group was constructed in [7] by Malliavin

Before the prove the existence of this limit we consider the process $(\mathbf{W}^z_{\psi,s})_t = \psi(t)+B^s_t$ where $\psi : [0,1] \to \mathbb{R}^m$ is a continuous function satisfying the condition $\psi(0) = 0$ and B^s_t is a multiple Brownian motion with the parameter s starting at the point z. The results obtained for this process will be applied for the further construction.

Some results for the process $(\mathbf{W}^z_{\psi,s})_t$. Let $\mathbb{W}^z_{\psi,s}$ denote the distribution of the process, $(\mathbf{W}^z_{\psi,s})_t$, $\mathbb{E}_{z,\psi,s}$ be the expectation relative to this distribution. We prove the existence of the weak limit of the measures in the right hand side of the equality below relative to the family of cylinder functions. More precisely we prove that the following limit exists

$$\int\limits_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{z,\psi,s}\{f(\omega)\,\mathbb{I}_{\{(\mathbf{W}^z_{\psi,s})_t\in U_\varepsilon(M)\}}\}}{\mathbb{W}^z_{\psi,s}\{(\mathbf{W}^z_{\psi,s})_t \in U_\varepsilon(M)\}}\ . \tag{2}$$

We find a function $\tilde{f} : \mathbb{R}^{k+1} \to \mathbb{R}$ and a finite number of points $\tau_1, \dots, \tau_k$, such that $f(\omega) = \tilde{f}(\omega(\tau_1), \dots, \omega(\tau_k), \omega(t))$.

$$\int\limits_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}\,\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega : \omega(t) \in U_\varepsilon(M)\}}$$
$$= \lim_{\varepsilon\to 0} \frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega+z+\psi)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M-z-\psi(t))\}}\,\mathbb{W}^0_{0,s}(d\omega)}{\mathbb{W}^0_{0,s}\{\omega : \omega(t) \in U_\varepsilon(M-z-\psi(t))\}}$$
$$= \lim_{\varepsilon\to 0} \frac{1}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t))} \cdot \int_{\mathbb{R}^m} \mathbb{P}^{\mathbb{W}}(\tau_1,0,dx_1) \int_{\mathbb{R}^m} \mathbb{P}^{\mathbb{W}}(\tau_2-\tau_1,x_1,dx_2)\dots$$
$$\int_{U_\varepsilon(M-\psi(t)-z)} \mathbb{P}^{\mathbb{W}}(t-\tau_k,x_k,dx_{k+1})\tilde{f}(x_1+z+\psi(\tau_1),\dots,x_k+z+\psi(\tau_k), x_{k+1}+z+\psi(t))$$

where $\mathbb{P}^{\mathbb{W}}(\tau,x,dz) = \frac{1}{(2\pi s\tau)^{\frac{m}{2}}}e^{\frac{|z-x|^2}{2s\tau}}dz$ is the distribution of a multiple Brownian motion with the parameter s. Since the function under the integral sign is bounded, by the Lebesgue theorem it suffices to prove the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\int\limits_{(U_\varepsilon(M-\psi(t)-z))} \tilde{f}(x_1+z+\psi(\tau_1),\dots,x_{k+1}+z+\psi(t))\mathbb{P}^{\mathbb{W}}(t-\tau_k,x_k,dx_{k+1})}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t))}$$
$$= \lim_{\varepsilon\to 0}\frac{\int\limits_{U_\varepsilon(M-\psi(t)-z-x_k)} \tilde{f}(x_1+z+\psi(\tau_1),\dots,x_{k+1}+x_k+z+\psi(t))\mathbb{P}^{\mathbb{W}}(t-\tau_k,0,dx_{k+1})}{\mathbb{P}^{\mathbb{W}}(t,0,U_\varepsilon(M-z-\psi(t))}.$$

Since we integrate relative to the variable x_{k+1} and the other variables are fixed, we can eliminate the other variables from our consideration when introducing the following notations: $g(x_{k+1}) = \tilde{f}(x_1 + z + \psi(\tau_1), \dots, x_{k+1} + x_k + z + \psi(t))$, $M_1 = M - \psi(t) - z - x_k$, $M_2 = M - z - \psi(t)$. Further, we rewrite this limit in the following way:

$$\lim_{\varepsilon\to 0} \frac{\frac{1}{(2\pi s(t-\tau_k))^{\frac{m}{2}}} \int_{U_\varepsilon(M_1)} e^{\frac{|x_k - x_{k+1}|^2}{2s(t-\tau_k)}} g(x_{k+1}) dx_{k+1}}{\frac{1}{(2\pi st)^{\frac{m}{2}}} \int_{U_\varepsilon(M_2)} e^{\frac{|x_{k+1}|^2}{2st}} dx_{k+1}} \ . \tag{3}$$

Let $\lambda_\varepsilon = \frac{1}{vol_{m-d}(\varepsilon)} \, l|_{U_\varepsilon(M_1)}$ and $\mu_\varepsilon = \frac{1}{vol_{m-d}(\varepsilon)} \, l|_{U_\varepsilon(M_2)}$, where l is the Lebesgue measure on $\mathbb{R}^m$. Then for the last limit we get

$$\lim_{\varepsilon\to 0} \frac{\int_{\mathbb{R}^m} g(x_{k+1}) \, e^{-\frac{|x_{k+1} - x_k|^2}{2s(t-\tau_k)}} \lambda_\varepsilon(dx_{k+1})}{\int_{\mathbb{R}^m} e^{-\frac{|x_{k+1}|^2}{2st}} \mu_\varepsilon(dx_{k+1})} \ .$$

Lemma 1. *The measures λ_ε and μ_ε converge weakly to the surface measures M_1 and M_2 respectively as ε tends to zero.*

Proof. Since M is a smooth compact Riemannian manifold (and hence, so are M_1 and M_2), we can find an ε-neighborhood of the manifold M where the normal spaces at each point of the manifold M do not have common points. Let g be an arbitrary uniformly bounded function $\mathbb{R}^m \to \mathbb{R}$. Pick a number ε such as discussed above and such that $|g(x) - g(y)| < \varepsilon$ for all $|x - y| < \varepsilon$. Further, denote by λ^ε and μ^ε (with the index ε on the top) measures on the manifolds M_1 and M_2 respectively, defined on Borelean subsets of the manifolds M_1 and M_2 in the following way: $\lambda^\varepsilon(B) = \lambda_\varepsilon(B^\varepsilon)$ where

$$B^\varepsilon = \{x + \sum_{i=1}^{m-d} t_i n_i(x), x \in B, n_i(x) \in N(x), |n_i| = 1, n_i \perp n_j \, (i \neq j), |t_i| \leqslant \varepsilon\}$$

and $N(x)$ denotes the normal space at the point x (the measure μ_ε is defined analogously on M_2). We prove the result only for the measure λ_ε omitting the index 1 for the manifold M_1 for simplicity of notations. Clearly, every point of the ε-neighborhood where the normal spaces do not intersect each other, can be uniquely presented as $x + tn_x$ where $n_x \in N(x)$. Define a new function $\bar{g}(x+tn_x) = g(x)$ for $x \in M$, $|t| < \varepsilon$. Obviously, for all $x \in U_\varepsilon(M)$ $|\bar{g}(x) - g(x)| < \varepsilon$. As ε tends to zero, the measures λ^ε converge to the surface measure λ_M on every Borelean subset of manifold M, and hence, weakly. We have

$$\int_{U_\varepsilon(M)} \bar{g}(x) \lambda_\varepsilon(dx) = \int_M g(x) \lambda^\varepsilon(dx) \to \int_M g(x) \lambda_M(dx) \ .$$

Note that

$$\Big| \int_{\mathbb{R}^m} g(x) \lambda_\varepsilon(dx) - \int_{U_\varepsilon(M)} \bar{g}(x) \lambda_\varepsilon(dx) \Big| < \varepsilon \lambda_\varepsilon(U_\varepsilon(M)) < K\varepsilon$$

where K is a constant. From this it follows

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^m} g(x)\lambda_\varepsilon(dx) = \int_M g(x)\lambda_M(dx)\ .$$

□

The proof of the existence of the limit (1). There exist a function $\tilde{f} : C([0,s],\mathbb{R}^m)^{k+1} \to \mathbb{R}$ and a finite number of points $\tau_1, \tau_2, \ldots, \tau_k$, such that $f(\omega) = \tilde{f}(\omega(\tau_1), \omega(\tau_2), \ldots, \omega(\tau_k), \omega(t))$. For this function we find a function $\tilde{\tilde{f}} : \mathbb{R}^{(k+1)\cdot(l+1)} \to \mathbb{R}$ and a finite number of points $\xi_1, \xi_2, \ldots, \xi_l$, such that $\tilde{f}(\omega(\tau_1), \omega(\tau_2), \ldots, \omega(\tau_k), \omega(t)) = \tilde{\tilde{f}}(\omega(\tau_1,\xi_1), \ldots, \omega(\tau_1,\xi_l), \ldots, \omega(t,\xi_l), \omega(t,s))$ (without loss of generality we can consider that the functions $\tilde{f}$ and $\tilde{\tilde{f}}$ depend on ω also at the points t and s). Taking into account that the process $\mathbf{W}_t$ is a Markov process (see [7]), we get

$$\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\tilde{\mathbb{W}}^y_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t,s)\in U_\varepsilon(M)\}}\,\tilde{\mathbb{W}}^y_\psi(d\omega)}{\tilde{\mathbb{W}}^y_\psi\{\omega : \omega(t,s)\in U_\varepsilon(M)\}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,s],\mathbb{R}^m)^t} f(\omega+y+\psi)\,\mathbb{I}_{\{\omega:\,\omega(t,s)\in U_\varepsilon(M-y(s)-\psi(t))\}}\tilde{\mathbb{W}}^0(d\omega)}{\tilde{\mathbb{W}}^0\{\omega : \omega(t,s)\in U_\varepsilon(M-y(s)-\psi(t))\}}$$

$$= \lim_{\varepsilon\to 0} \frac{\int\limits_{C([0,s],\mathbb{R}^m)} \mathbb{P}^{\tilde{\mathbb{W}}}(\tau_1, 0, dw_1) \quad \int\limits_{C([0,s],\mathbb{R}^m)} \mathbb{P}^{\tilde{\mathbb{W}}}(\tau_2-\tau_1, w_1, dw_2)}{\tilde{\mathbb{W}}^0\circ\pi_s^{-1}\circ\pi_t^{-1}(U_\varepsilon(M-y(s)-\psi(t)))} \cdots$$

$$\int_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)))} \mathbb{P}^{\tilde{\mathbb{W}}}(t-\tau_k, w_k, dw_{k+1})$$
$$\tilde{f}\big(w_1+y+\psi(\tau_1), \ldots, w_k+y+\psi(\tau_k), w_{k+1}+y+\psi(t)\big)$$

Since the function under the integral sign is bounded, it suffices to prove the existence of the limit as ε goes to 0, for the following quantity

$$\frac{\int\limits_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)))} \tilde{f}(w_1+y+\psi(\tau_1), \ldots, w_{k+1}+y+\psi(t))\mathbb{P}^{\tilde{\mathbb{W}}}(t-\tau_k, w_k, dw_{k+1})}{\tilde{\mathbb{W}}^0\circ\pi_s^{-1}\circ\pi_t^{-1}(U_\varepsilon(M-y(s)-\psi(t)))}$$

$$= \frac{\int\limits_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)-w_k(s)))} \tilde{f}(w_1+y+\psi(\tau_1), \ldots, w_{k+1}+w_k+y+\psi(t))\mathbb{P}^{\tilde{\mathbb{W}}}(t-\tau_k, 0, dw_{k+1})}{\tilde{\mathbb{W}}^0\circ\pi_s^{-1}\circ\pi_t^{-1}(U_\varepsilon(M-y(s)-\psi(t)))}$$

Since the integration variable in the last integral is the variable w_{k+1} then for simplicity we introduce a function g expressing dependence on w_{k+1} only. More precisely the function g is such that $\tilde{f}(w_1+y+\psi(\tau_1),\dots,w_{k+1}+w_k+y+\psi(t)) = g(w_{k+1}(\xi_1),\dots,w_{k+1}(\xi_l),w_{k+1}(s))$. Proceeding with the above calculation we get

$$\frac{l(\xi_i,\tau_k,s,t)\int\limits_{\mathbb{R}^m} e^{-\frac{|x_1|^2}{2\xi_1(t-\tau_k)}}dx_1\dots \int\limits_{U_\varepsilon(M-\psi(t)-y(s)-w_k(s))} g(x_1,\dots,x_{l+1})\, e^{-\frac{|x_{l+1}-x_l|^2}{2(s-\xi_l)(t-\tau_k)}}dx_{l+1}}{\frac{1}{(2\pi st)^{\frac{m}{2}}}\int\limits_{U_\varepsilon(M-\psi(t)-y(s))} e^{-\frac{|x_1|^2}{2st}}dx_1}$$

where $l(\xi_i,\tau_k,s,t) = \frac{1}{(2\pi(t-\tau_k))^{\frac{(l+1)m}{2}}}\frac{1}{(\xi_1(\xi_2-\xi_1)\cdots(s-\xi_l))^{\frac{m}{2}}}$. And now we need to prove the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\int_{\mathbb{R}^m}\mathbb{I}_{U_\varepsilon(M-\psi(t)-y(s)-w_k(s))}\, g(x_1,\dots,x_{l+1})\, e^{-\frac{|x_{l+1}-x_l|^2}{2(s-\xi_l)(t-\tau_k)}}dx_{l+1}}{\int_{\mathbb{R}^m}\mathbb{I}_{U_\varepsilon(M-\psi(t)-y(s))}\, e^{-\frac{|x_1|^2}{2st}}dx_1}\ .$$

This limit has the form (3) and its existence is proved above. Hence the existence of the limit (1) is proved for the family of cylinder functions.

2 Asymptotics for integrals of Gaussian type

PROPOSITION 1. *Let $\imath$ be an isometric imbedding of the manifold M into $\mathbb{R}^m$, $g\in C^2(M)$. Then*

$$\frac{1}{(2\pi t)^{\frac{d}{2}}}\int_M g(z)e^{-\frac{|z-y|^2}{2t}}\lambda_M(dz) = g(y)+\frac{t}{8}\,g(y)\big(c(y)-scal(y)\big)-\frac{t}{2}\Delta_M g(y)+tR(t,y), \tag{4}$$

where $|R(t,y)| < Kt^{1/2}$ and K is a constant which is independent of y; the function $c(y)$ has the form:

$$c(y) = \sum_{i,j}\sum_{\alpha}\Big(\frac{\partial^2 i^\alpha}{\partial x^i\partial x^j}\Big)^2(0)\ , \tag{5}$$

where x^i are normal coordinates in a neighborhood U_y of the point y provided by a homeomorphism of U_y on a neighborhood U of the point zero in $\mathbb{R}^d$, i is the imbedding $\imath$ written in local coordinates x^i. Independently of local coordinates, $c(y)$ can be written as

$$c(y) = -\frac{1}{2}\Delta_M\Delta_M\ |y-\ \cdot|^2\big|_y - \frac{1}{3}\,scal(y)\ , \tag{6}$$

and hence, $c(y)$ depends only on the imbedding $\imath$.

Proof. It is well known that $|\imath(z)-\imath(y)|^2 = d(y,z)^2+\varphi(y,z)$ where d is a geodesic distance in M and $\varphi(y,z) = O(d(y,z)^4)$. Let $\psi_y : U \to U_y$ be the diffeomorphism providing the normal coordinates in U_y, $f_y(x) = \varphi(y,\psi_y(x))$, $h_y(x) = \sqrt{\det g_{ij}(x)}\, g(\psi_y(x))$ where g_{ij} is the metric tensor. We have

$$\int_{U_y} e^{-\frac{|z-y|^2}{2t}} g(z)\lambda_M(dz) = \int_{U_y} e^{-\frac{d(y,z)^2+\varphi(y,z)}{2t}} g(z)\lambda_M(dz) = \int_U e^{-\frac{|x|^2+f_y(x)}{2t}} h_y(x)dx \ .$$

By results of [4]

$$\frac{1}{(2\pi t)^{\frac{d}{2}}}\int_U e^{-\frac{|x|^2+f_y(x)}{2t}} h_y(x)dx = h_y(0)+\frac{t}{2}\,\Delta h_y(0) - \frac{t}{16}\,h_y(0)\Delta\Delta f_y(0)+\bar R_t \quad (7)$$

where $|\bar R_t| < \bar K t^{1/2}$, and $\bar K$ is a constant. As it was mentioned in [4] the sets $\{\varphi(y,\cdot) : y\in M\}$ and $\{g_{ij}(y) : y \in M\}$ are uniformly in y bounded in $C^5(M)$ and $C^3(U)$ respectively, and the neighborhood U can be chosen independently of y. This implies that the constant $\bar K$ can be also chosen independently of y. Note that $h_y(0) = g(y)$, and as it was calculated in [4] $\Delta h_y(0) = -\Delta_M u(y) - \frac{1}{3}u(y)scal(y)$. Calculate $\Delta\Delta f_y(0)$. Note that $\Delta\Delta\, d(y,\psi_y(x))^2 = \Delta\Delta|x|^2 = 0$, and hence $\Delta\Delta f_y(0) = \Delta\Delta\big(|\imath\circ\psi_y(x)-\imath(y)|^2\big)|_{x=0}$. We have denoted $\imath\circ\psi_y$ as i. Taking into account this we get

$$\Delta\Delta f_y(0) = \Delta\Delta\sum_\alpha (i^\alpha(x)-i^\alpha(0))^2|_{x=0} \ .$$

It is easy to see that the result of the calculation of the right-hand side does not depend on $i(0)$, so we may assume that $i(0)=0$. It is well known (see for example [15]) that

$$g_{ij}(x) = \sum_\alpha \frac{\partial i^\alpha}{\partial x^i}\frac{\partial i^\alpha}{\partial x^j}(x) \ .$$

From this we get

$$\sum_{i,j}\frac{\partial^2 g_{ij}}{\partial x^i\partial x^j} = \sum_{i,j}\sum_\alpha\Big(\frac{\partial^2 i^\alpha}{\partial x^i\partial x^j}\Big)^2 + 2\sum_{i,j}\sum_\alpha \frac{\partial i^\alpha}{\partial x^j}\frac{\partial^3 i^\alpha}{(\partial x^i)^2\partial x^j} + \sum_{i,j}\sum_\alpha \frac{\partial^2 i^\alpha}{(\partial x^i)^2}\frac{\partial^2 i^\alpha}{(\partial x^j)^2} \ . \quad (8)$$

Further

$$\sum_\alpha \frac{\partial^2 (i^\alpha(x))^2}{(\partial x^j)^2} = 2\sum_\alpha\Big(\frac{\partial i^\alpha(x)}{\partial x^j}\Big)^2 + i^\alpha(x)\frac{\partial^2 i^\alpha(x)}{(\partial x^j)^2} = 2\,g_{jj} + 2\sum_\alpha i^\alpha(x)\frac{\partial^2 i^\alpha(x)}{(\partial x^j)^2} \ .$$

Differentiating the last equality twice in x^i, taking the sum in i and j, and using (8) we get

$$\Delta\Delta\sum_\alpha (i^\alpha(x))^2|_{x=0} = 2\Big(\sum_{i,j}\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) + 2\sum_{i,j}\sum_\alpha \frac{\partial i^\alpha}{\partial x^j}\frac{\partial^3 i^\alpha}{(\partial x^i)^2\partial x^j}(0)$$
$$+\sum_{i,j}\sum_\alpha \frac{\partial^2 i^\alpha}{(\partial x^i)^2}\frac{\partial^2 i^\alpha}{(\partial x^j)^2}(0)\Big)$$
$$= 2\Big(\sum_{i,j}\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) + \sum_{i,j}\frac{\partial^2 g_{ij}}{\partial x^i\partial x^j}(0) - \sum_{i,j}\sum_\alpha\Big(\frac{\partial^2 i^\alpha}{\partial x^i\partial x^j}\Big)^2(0)\Big).$$

In normal coordinates the metric tensor has the form [14]

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{iklj}x^k x^l + O(|x|^4)$$

where R_{iklj} is the curvature tensor. From this we have

$$\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) = \frac{2}{3}R_{jiij}(0) = -\frac{2}{3}R_{ijij}(0)\ , \quad \frac{\partial^2 g_{ij}}{\partial x^i\partial x^j}(0) = \frac{1}{3}(R_{iijj}(0) + R_{ijij}(0))\ .$$

In normal coordinates $R_{ijij}(x) = scal(x)$ (see [15]). The scalar curvature does not depend on choice of local coordinates, and hence it only depends on a point of the manifold. Taking into account this, we get

$$\sum_{i,j}\frac{\partial^2 g_{jj}}{(\partial x^i)^2}(0) + \sum_{i,j}\frac{\partial^2 g_{ij}}{\partial x^i\partial x^j}(0) = -\frac{1}{3}scal(y)\ .$$

Finally

$$\Delta\Delta f_y(0) = \Delta\Delta\sum_\alpha (i^\alpha(x))^2|_{x=0} = -2(\tfrac{1}{3}scal(y) + c(y))$$

where $c(y)$ is defined by (5). This implies also (6) if we take into account that the operator Δ_M in normal coordinates becomes the opposite to the ordinary Laplace operator. Choose now neighborhoods U_y of the form: $U_y = \{z \in M : |z-y| < \varepsilon\}$ where ε is chosen as follows. As it was mentioned in [4] the diameter of the neighborhood of the point y where we can introduce the normal coordinate system, bounded from zero, say by ε. Define each U_y with the help of this ε. We have

$$\frac{1}{(2\pi t)^{\frac{d}{2}}}\int_{M\setminus U_y} g(z)e^{-\frac{|z-y|^2}{2t}}\lambda_M(dz) \leqslant \frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{\varepsilon^2}{2t}}\int_M |g(z)|\lambda_M(dz) < t^{3/2} \qquad (9)$$

for t smaller than some number t_0. The last estimate holds uniformly in y for all $t < t_0$. Substituting the expressions for $\Delta h_y(0)$ and $\Delta\Delta f_y(0)$ into (7) and taking into account (9) we get (4) with $R(t,y)$ satisfying $|R(t,y)| < Kt^{1/2}$, where the constant K is independent of y. □

Corollary 1. *Let $g \in C^2(M)$. Then we have the following asymptotic:*

$$\frac{\int_M g(z) e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz)} = g(y) - \frac{t}{2}\Delta_M g(y) + tR_1(t,y) \ , \tag{10}$$

where $|R_1(t,y)| < K_1 t^{1/2}$ and K_1 is a constant which is independent of y.

Proof. The statement of the corollary easily follows from the the above proposition applied to the functions $g(y)$ and $g(y) \equiv 1$ respectively. □

Proposition 2. *Let $\imath$ be an isometric imbedding of the manifold M into $\mathbb{R}^m$, $g \in C^2(M)$, $y \in M$, $0 < t < t_1 < 1$, u_1 and u_2 are such that $|u_2 - u_1| < t_1^{\alpha}$, $\alpha > 0$. Further let Pr_M be mapping of projection onto the manifold M along the subspaces normal to the manifold which are defined in the proper neighborhood of the manifold, $(u_2-u_1)^y_M$ and $(u_2-u_1)^y_{\perp}$ are such that:*

$$(u_2-u_1)^y_M = \mathrm{Pr}_M(y+u_2-u_1) - y, \quad (u_2-u_1)^y_{\perp} = y+u_2-u_1-\mathrm{Pr}_M(y+u_2-u_1).$$

Then the following asymptotic holds:

$$\frac{\int_M g(z) e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)} = g(y+(u_2-u_1)^y_M) - \frac{t}{2}\Delta_M g(y+(u_2-u_1)^y_M) \\ + g(y+(u_2-u_1)^y_M)\mathcal{R}(y, u_2-u_1) + tR_2(t,t_1,y,u_2-u_1); \tag{11}$$

for the rest terms we have: $|R_2(t,t_1,y,u_2-u_1)| < K_2 t_1^{\alpha}$ (K_2 is a constant),

$$\mathcal{R}(y,u_2-u_1) = -\frac{1}{4}(\Delta_M \imath(y+(u_2-u_1)^y_M), (u_2-u_1)^y_{\perp})^2 \\ + \sum_{n=3}^{N} \sum k(n) \prod_{l_1+\cdots+l_s=n} D^{2,l_i}_M(\imath(\cdot), (u_2-u_1)^y_{\perp})^{l_i}\big(y+(u_2-u_1)^y_M\big), \tag{12}$$

where D^{2,l_i}_M are differential operators on M having the following form $D^{2,l_i}_M = (\wedge_{k=1}^{l_i}\nabla_M^{(i_k)}, \wedge_{k=1}^{l_i}\nabla_M^{(j_k)})$; the operators are applied to a product of l_i functions, i_k and j_k are numbers from 1 till l_i which have the meaning of the function number in the product on which the corresponding operator acts; the index 2 says that i_k and j_k take the same values exactly 2 times, $k(n)$ are rational functions; the second sum in the last term contains a finite number of items, the number N is chosen so that $t_1^{(N-1)\alpha} < t$.

Proof. We have

$$\frac{\int_M g(z)\, e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-(u_2-u_1)|^2}{2t}} \lambda_M(dz)} = \frac{\int_M g(z)\, e^{-\frac{|z-(y+(u_2-u_1)^y_M)-(u_2-u_1)^y_\perp|^2}{2t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)-(u_2-u_1)^y_\perp|^2}{2t}} \lambda_M(dz)}$$
$$= \frac{\int_M g(z)\, e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}\, e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}} \lambda_M(dz)}{\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}\, e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}} \lambda_M(dz)}$$

Everywhere below K, R, $\mathcal{K}$ are some constants (not necessarily equal). We choose an ε-neighborhood $U_{y+(u_2-u_1)^y_M}$ of the point $y+(u_2-u_1)^y_M$, and fix there a system of normal coordinates. We have

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int\limits_{M\setminus U_{y+(u_2-u_1)^y_M}} g(z)\, e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}\, e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}} \lambda_M(dz) < \frac{K}{(2\pi t)^{\frac{d}{2}}}\, e^{\left(\frac{R}{t\, t_1^{-\alpha}} - \frac{\varepsilon^2}{2t}\right)} .$$

Let h_y and f_y denote the same as in the proof of the Proposition 1 but relative to the point $y+(u_2-u_1)^y_M$, more precisely $f_y(x) = \varphi(y+(u_2-u_1)^y_M, \psi_y(x))$, $h_y(x) = \sqrt{\det g_{ij}(x)}\, g(\psi_y(x))$, where $\psi_y : U \to U_{y+(u_2-u_1)^y_M}$ is a homeomorphism providing normal coordinates x_k in the neighborhood $U_{y+(u_2-u_1)^y_M}$, g_{ij} is the metric tensor. Further we will calculate the integrals in the numerator and denominator relative to the system of normal coordinates x_k in the neighborhood $U_{y+(u_2-u_1)^y_M}$ of the point $y+(u_2-u_1)^y_M$. Note that $z-(y+(u_2-u_1)^y_M) = x+\tilde f(x)$ where x belongs to $T_{y+(u_2-u_1)^y_M}$ - the tangent space to the manifold M at the point $y+(u_2-u_1)^y_M$, $\tilde f(x)$ is orthogonal to $T_{y+(u_2-u_1)^y_M}$ and such that $\tilde f(0)=0$, $\frac{\partial \tilde f}{\partial x_k}(0) = 0$ for all k. Since $(u_2-u_1)^y_\perp$ is orthogonal to $T_{y+(u_2-u_1)^y_M}$, we have

$$\begin{aligned}(z-(y+(u_2-u_1)^y_M), (u_2-u_1)^y_\perp) &= (\tilde f(x), (u_2-u_1)^y_\perp) \\ = \frac{1}{2}\Big(\frac{\partial^2 \tilde f}{\partial x^i \partial x^j}, (u_2-u_1)^y_\perp\Big) x^i x^j &+ \frac{1}{6}\Big(\frac{\partial^3 \tilde f}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_3}}, (u_2-u_1)^y_\perp\Big) x^{i_1}x^{i_2}x^{i_3} \\ &+ \|\tilde f\|_4\, \theta(x)\, |x|^4\, t_1^{\alpha} .\end{aligned}$$

Let ν_t denote the Gaussian distribution with the covariance matrix tI. Let t be sufficiently small such that for all $0 \leqslant k \leqslant 3N$

$$\int_{\mathbb{R}^d\setminus U} |x|^k \nu_t(dx) < t^N . \tag{13}$$

Further, let θ_i denote functions satisfying the condition $|\theta_i| \leqslant 1$.

We have

$$
\begin{aligned}
e^{\frac{(\tilde f(x),(u_2-u_1)^y_\perp)}{t}} &= 1+\frac{1}{2}\Big(\frac{\partial^2\tilde f}{\partial x^i\partial x^j}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^ix^j}{t}\\
&+\frac{1}{6}\Big(\frac{\partial^3\tilde f}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_3}}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}x^{i_2}x^{i_3}}{t}\\
&+\sum_{n=2}^{N-1}\frac{1}{2^n n!}\prod_{s=1}^{n}\Big(\frac{\partial^2\tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n}}}{t^n}\\
&+\sum_{n=2}^{N-1}\frac{1}{3\cdot 2^n n!}\prod_{s=1}^{n-1}\Big(\frac{\partial^2\tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}\,,(u_2-u_1)^y_\perp\Big)\\
&\qquad\times\Big(\frac{\partial^3\tilde f}{\partial x^{i_{2n-1}}\partial x^{i_{2n}}\partial x^{i_{2n+1}}}\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n+1}}}{t^n}\\
&+\|\tilde f\|_4\,\theta(x)\,\frac{|x|^4}{t}\,t_1^\alpha+\|\tilde f\|_3^N\,t\cdot t_1^\alpha\textstyle\sum_{s=0}^{N}k(s,N)\theta_s(x)\dfrac{|x|^{2N+s}}{t^N}\;.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(2\pi t)^{\frac{m}{2}}}\int_{\mathbb{R}^d}\exp\Big(-\frac{|x|^2+f_y(x)}{2t}\Big)h_y(x)\,e^{\frac{(\tilde f(x),(u_2-u_1)^y_\perp)}{t}}dx\\
&\quad=\int_{\mathbb{R}^d}\Big(\Big(1-\frac{1}{48t}\,\frac{\partial^4 f_y}{\partial x^{i_1}\cdots\partial x^{i_4}}(0)\,x^{i_1}\cdots x^{i_4}+\frac{\|f_y\|_5\theta_1(x)|x|^5}{t}\Big)\\
&\quad\times\Big(h_y(0)+\frac{\partial h_y}{\partial x^{i_1}}(0)x^{i_1}+\frac{1}{2}\,\frac{\partial^2 h_y}{\partial x^{i_1}\partial x^{i_2}}(0)\,x^{i_1}x^{i_2}+\|h_y\|_3\,\theta_2(x)\,|x|^3\Big)\\
&\quad\times\Big(1+\sum_{n=1}^{N-1}\frac{1}{2^n n!}\prod_{s=1}^{n}\Big(\frac{\partial^2\tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}(0)\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n}}}{t^n}\\
&\qquad+\sum_{n=1}^{N-1}\frac{1}{3\cdot 2^n n!}\prod_{s=1}^{n-1}\Big(\frac{\partial^2\tilde f}{\partial x^{i_{2s-1}}\partial x^{i_{2s}}}(0)\,,(u_2-u_1)^y_\perp\Big)\\
&\qquad\times\Big(\frac{\partial^3\tilde f}{\partial x^{i_{2n-1}}\partial x^{i_{2n}}\partial x^{i_{2n+1}}}(0)\,,(u_2-u_1)^y_\perp\Big)\,\frac{x^{i_1}\cdots x^{i_{2n+1}}}{t^n}\\
&\quad+\|\tilde f\|_4\,\theta(x)\,\frac{|x|^4}{t}\,t_1^\alpha+\|\tilde f\|_3^N\,t\cdot t_1^\alpha\textstyle\sum_{s=0}^{N}k(s,N)\theta_s(x)\dfrac{|x|^{2N+s}}{t^N}\Big)\nu_t(dx)
\end{aligned}
\tag{14}
$$

The number N is chosen so that $t_1^{\alpha N}<t\,t_1^\alpha$. The norms $\|f\|_k$ introduced as in [4]. By condition (13), replacing the integration over the whole space $\mathbb{R}^d$ with the integration over the neighborhood U gives the rest term which is not bigger than $K\,t\,t_1^\alpha$.

Performing the integrating in (14) we get

$$\frac{1}{(2\pi t)^{\frac{m}{2}}}\int_{\mathbb{R}^d}\exp\Big(-\frac{|x|^2+f_y(x)}{2t}\Big)h_y(x)\,e^{\frac{(\tilde f(x),(u_2-u_1)^y_\perp)}{t}}dx$$

$$= h_y(0)+\frac{t}{2}\Delta h_y(0)-\frac{t}{16}\,h_y(0)\Delta\Delta f_y(0)+\frac{1}{2}(\Delta\tilde f(0),(u_2-u_1)^y_\perp)$$

$$+\sum_{n=2}^{N-1}\frac{1}{2^n n!}\sum\prod_{l_1+\cdots+l_s=n}D^{2,l_i}(\tilde f(\cdot),(u_2-u_1)^y_\perp)^{l_i}(0)+\tilde R(t,t_1). \quad (15)$$

Here D^{2,l_i} are differential operators of the form $D^{2,l_i}=(\wedge_{k=1}^{l_i}\nabla^{(i_k)},\wedge_{k=1}^{l_i}\nabla^{(j_k)})$; the operators are applied to the product of l_i functions, i_k and j_k are numbers from 1 till l_i having the meaning of the number of the function in the product on which the corresponding operator acts, the index 2 says that the indexes i_k and j_k take the same value exactly 2 times; the second sum in the last term contains a finite number of terms, $\tilde R(t,t_1)$ is the rest term which we estimate later. Note that $\Delta\tilde f(0)=\Delta_M\imath(y+(u_2-u_1)^y_M)$, $D^{2,l_i}(\tilde f(\cdot),(u_2-u_1)^y_\perp)^{l_i}(0)=D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}(y+(u_2-u_1)^y_M)$. Further, applying the Proposition 1 we get

$$\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}g(z)\,e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz)=g(y+(u_2-u_1)^y_M)$$

$$-\frac{t}{2}\Delta_M g(y+(u_2-u_1)^y_M)+\frac{t}{8}\,g(y+(u_2-u_1)^y_M)$$

$$\times\big(c(y+(u_2-u_1)^y_M)-scal(y+(u_2-u_1)^y_M)\big)-\frac{1}{2}(\Delta_M\imath(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)$$

$$+\sum_{n=2}^{N-1}\frac{1}{2^n n!}\sum\prod_{l_1+\cdots+l_s=n}D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}(y+(u_2-u_1)^y_M)+\tilde R(t,t_1).$$

Finally we get

$$\frac{\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}g(z)\,e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz)}{\int_M e^{-\frac{|z-(y+(u_2-u_1)^y_M)|^2}{2t}}\,e^{\frac{(z-(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)}{t}}\lambda_M(dz)}$$

$$=g(y+(u_2-u_1)^y_M)\big(1-\frac{1}{4}(\Delta_M\imath(y+(u_2-u_1)^y_M),(u_2-u_1)^y_\perp)^2$$

$$+\sum_{n=3}^{N-1}\sum k(n)\prod_{l_1+\cdots+l_s=n}D^{2,l_i}_M(\imath(\cdot),(u_2-u_1)^y_\perp)^{l_i}(y+(u_2-u_1)^y_M)\big)$$

$$-\frac{t}{2}\Delta_M g(y+(u_2-u_1)^y_M)+R(t,t_1),$$

$k(n)$ are rational functions, $R(t,t_1)$ is the rest term; by the equality

$$\int_{\mathbb{R}^d} |x|^{2N+s} \nu_t(dx) = \mathcal{K}\, t^{N+\frac{s}{2}},$$

$R(t,t_1)$ is not bigger than $K\, t\, t_1^{\alpha}$. □

3 The second step of construction $\mathbf{W}_M^x$

Let $\mathcal{P}_1 = \{t_0 = 0 \leqslant t_1 \leqslant \ldots \leqslant t_n = 1\}$ be a partition of the interval $[0,1]$, $\varphi : [0,1] \to M - x$ a continuous function such that $\varphi(0) = 0$. If E is a LCS then every $\omega \in E^1$ can be identified with the n-tuple $(\omega_1, \omega_2, \ldots, \omega_n) \in E^{t_1} \times E^{t_2-t_1} \times \ldots \times E^{t_n - t_{n-1}}$, where ω_j is defined on the interval $[0, t_j - t_{j-1}]$ by $\omega_j(t) = \omega(t_{j-1} + t)$. Define the function $\varphi_{t_{i-1}t_i}$ on the interval $[0, t_i - t_{i-1}]$ by

$$\varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1} + t) - \varphi(t_{i-1}) \ .$$

Define the measures $\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}$ and $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ by the formulas

$$\int\limits_{C([0,1],\mathbb{R}^m)} h(\omega) \mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,t_1],\mathbb{R}^m)} \mathbb{W}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int\limits_{C([0,t_2-t_1],\mathbb{R}^m)} \mathbb{W}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2) \ldots$$
$$\int\limits_{C([0,t_n-t_{n-1}],\mathbb{R}^m)} \mathbb{W}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) h(\omega_1, \omega_2, \ldots, \omega_n) \ .$$

$$\int\limits_{C([0,s],\mathbb{R}^m)^1} f(\omega) \tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) = \int\limits_{C([0,s],\mathbb{R}^m)^{t_1}} \tilde{\mathbb{W}}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int\limits_{C([0,s],\mathbb{R}^m)^{t_2-t_1}} \tilde{\mathbb{W}}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2) \ldots$$
$$\int\limits_{C([0,s],\mathbb{R}^m)^{t_n-t_{n-1}}} \tilde{\mathbb{W}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) f(\omega_1, \omega_2, \ldots, \omega_n) \ . \tag{16}$$

The measure $\tilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ is well-defined. To show this we should check that every function $\varphi_{t_{i-1}t_i}$ is such that $\varphi_{t_{i-1}t_i}(0) = 0$, which obviously holds, and every function $\omega_i(t_i - t_{i-1})$ is such that $\omega_i(t_i - t_{i-1})(0) \in M$. Check the last assertion. We have

$$\omega_1(t_1)(0) = \omega(t_1)(0) = x + \varphi(t_1) \in M;$$
$$\omega_2(t_2 - t_1)(0) = \omega_1(t_1)(0) + \varphi_{t_1t_2}(t_2 - t_1)$$
$$= x + \varphi(t_1) + \varphi(t_2) - \varphi(t_1) = x + \varphi(t_2) \in M;$$
$$\ldots$$
$$\omega_i(t_i - t_{i-1})(0) = \omega_{i-1}(t_{i-1} - t_{i-2})(0) + \varphi_{t_it_{i-1}}(t_i - t_{i-1}) = x + \varphi(t_i) \in M.$$

4 The third step of construction $\mathbf{W}_M^x$

Now we change the roles of s and t so that s will be a time parameter and t will be a variable of a function from the state space. Since s is the time parameter we rewrite the process $\mathbf{W}_{M,\varphi,s}^x$ as $\mathbf{W}_{M,s}^{x+\varphi}$. The function $x+\varphi$ is the value of the process at the moment 0. In the following it will be rather convenient to consider that $\varphi(0)=x$ and denote the above process as $\mathbf{W}_{M,s}^{\varphi}$. The distribution of the process $\mathbf{W}_{M,s}^{\varphi}$ we denote by $\mathbb{W}_{M,s}^{\varphi}$.

Let $\mathcal{P}_2=\{s_0=0\leqslant s_1,\ldots\leqslant s_k=1\}$ be a partition of the interval $[0,1]$. The time parameter during the following construction is the parameter s. We define a measure $\mathbb{W}_{M,\mathcal{P}_1,\mathcal{P}_2}^x$ by

$$\int\limits_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}_{M,\mathcal{P}_1,\mathcal{P}_2}^x(d\omega)=\int\limits_{C([0,1],\mathbb{R}^m)^{s_1}}\tilde{\mathbb{W}}_{M,s_1,\mathcal{P}_1}^x(d\omega_1)\int\limits_{C([0,1],\mathbb{R}^m)^{s_2-s_1}}\tilde{\mathbb{W}}_{M,s_2-s_1,\mathcal{P}_1}^{\omega_1(s_1)}(d\omega_2)\ldots \int\limits_{C([0,1],\mathbb{R}^m)^{s_n-s_{n-1}}}\tilde{\mathbb{W}}_{M,s_n-s_{n-1},\mathcal{P}_1}^{\omega_{n-1}(s_{n-1}-s_{n-2})}(d\omega_n)f(\omega_1,\ldots,\omega_n)\ . \tag{17}$$

THEOREM 1. *For every $x\in M$, as the meshes of $\mathcal{P}_1$ and $\mathcal{P}_2$ tend to zero, the sequence of measures $\mathbb{W}_{M,\mathcal{P}_1,\mathcal{P}_2}^x$ converges weakly relative to the family of cylinder functions to the measure $\mathbb{W}_M^x$. The measure $\mathbb{W}_M^x$, being considered as a distribution of a process with values in $C([0,1],M)$, possesses the transition density at the moment t which coincides with the distribution of a multiple Brownian motion on the manifold M with parameter t starting at the point x.*

Proof. Consider the integral

$$\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\,\mathbb{W}_{M,\psi,s,t}^z(d\omega)\ ,$$

where $z\in M$, $\psi(0)=0$ and the function g is such that there exists a function

$\tilde g:\mathbb{R}^m\to\mathbb{R}$ and $g(\omega)=\tilde g(\omega(t))$. From (2) it follows that

$$\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\,\mathbb{W}^z_{M,\psi,s,t}(d\omega)=\lim_{\varepsilon\to 0}\frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}(\omega)\,\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}$$
$$=\lim_{\varepsilon\to 0}\frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega+\psi)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}(\omega+\psi)\,\mathbb{W}^z_{0,s}(d\omega)}{\mathbb{W}^z_{0,s}\{\omega:\,\omega(t)\in U_\varepsilon(M)-\psi(t)\}}$$
$$=\lim_{\varepsilon\to 0}\frac{\int\limits_{C([0,t],\mathbb{R}^m)} g(\omega)\,\mathbb{I}_{\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}(\omega)\mathbb{W}^{z+\psi(t)}_{0,s}(d\omega)}{\mathbb{W}^{z+\psi(t)}_{0,s}\{\omega:\,\omega(t)\in U_\varepsilon(M)\}}$$
$$=\lim_{\varepsilon\to 0}\frac{\int\limits_{U_\varepsilon(M)}\tilde g(x_1)\mathbb{W}^{z+\psi(t)}_{0,s}\circ\pi_t^{-1}(dx_1)}{\mathbb{W}^{z+\psi(t)}_{0,s}\circ\pi_t^{-1}\big(U_\varepsilon(M)\big)}$$
$$=\frac{\int\limits_M e^{-\frac{|x_1-z-\psi(t)|^2}{2ts}}\,\tilde g(x_1)\,\lambda_M(dx_1)}{\int\limits_M e^{-\frac{|x_1-z-\psi(t)|^2}{2ts}}\,\lambda_M(dx_1)}\,. \tag{18}$$

Now we pass to the calculation of the integral (17). It suffices to prove its convergence when the function f is such that $f(\omega)=\tilde f(\omega(s))$ for some function $\tilde f: C([0,1],\mathbb{R}^m)\to\mathbb{R}$. We have

$$\int\limits_{C([0,s],\mathbb{R}^m)^1} f(\omega)\,\tilde{\mathbb{W}}^\varphi_{M,s,\mathcal{P}_1}(d\omega)=\int\limits_{C([0,1],\mathbb{R}^m)}\tilde f(w)\,\tilde{\mathbb{W}}^\varphi_{M,s,\mathcal{P}_1}\circ\pi_s^{-1}(dw)=\int\limits_{C([0,1],\mathbb{R}^m)}\tilde f(w)\,\mathbb{W}^\varphi_{M,s,\mathcal{P}_1}(dw).$$

From this we get

$$\int\limits_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)=\int\limits_{C([0,1],\mathbb{R}^m)}\tilde{\mathbb{W}}^x_{M,s_1,\mathcal{P}_1}\circ\pi_{s_1}^{-1}(dw_1)\int\limits_{C([0,1],\mathbb{R}^m)}\tilde{\mathbb{W}}^{w_1}_{M,s_2-s_1,\mathcal{P}_1}\circ\pi_{s_2-s_1}^{-1}(dw_2)\ldots$$
$$\int\limits_{C([0,1],\mathbb{R}^m)}\tilde{\mathbb{W}}^{w_{n-1}}_{M,s_n-s_{n-1},\mathcal{P}_1}\circ\pi_{s_n-s_{n-1}}^{-1}(dw_n)\tilde f(w_n)$$
$$=\int\limits_{C([0,1],\mathbb{R}^m)}\mathbb{W}^x_{M,s_1,\mathcal{P}_1}(dw_1)\int\limits_{C([0,1],\mathbb{R}^m)}\mathbb{W}^{w_1}_{M,s_2-s_1,\mathcal{P}_1}(dw_2)\ldots$$
$$\int\limits_{C([0,1],\mathbb{R}^m)}\mathbb{W}^{w_{n-2}}_{M,s_{n-1}-s_{n-2},\mathcal{P}_1}(dw_{n-1})\int\limits_{C([0,1],\mathbb{R}^m)}\mathbb{W}^{w_{n-1}}_{M,s_n-s_{n-1},\mathcal{P}_1}(dw_n)\tilde f(w_n).$$

Let us first assume that the function f is such that there exist a function $p:\mathbb{R}^m\to\mathbb{R}$ and numbers $t,s\in[0,1]$ such that $f(\omega)=p(\omega(t,s))$. Taking into account that each of these integrals has the form (18) we obtain that $\int_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)$ has the following form:

$$\frac{\int_M e^{-\frac{|x_1-x|^2}{2\Delta s_1\Delta t_1}}dx_1}{\int_M e^{-\frac{|\bar{x}_1-x|^2}{2\Delta s_1\Delta t_1}}d\bar{x}_1}\cdots\frac{\int_M e^{-\frac{|x_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}}dx_{n-1}}{\int_M e^{-\frac{|\bar{x}_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}}d\bar{x}_{n-1}}\frac{\int_M e^{-\frac{|x_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}}dx_n}{\int_M e^{-\frac{|\bar{x}_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}}d\bar{x}_n}$$

$$\frac{\int_M e^{-\frac{|y_1-x_1|^2}{2\Delta s_2\Delta t_1}}dy_1}{\int_M e^{-\frac{|\bar{y}_1-x_1|^2}{2\Delta s_2\Delta t_1}}d\bar{y}_1}\cdots\frac{\int_M e^{-\frac{|y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}}dy_{n-1}}{\int_M e^{-\frac{|\bar{y}_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}}d\bar{y}_{n-1}}\frac{\int_M e^{-\frac{|y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}}dy_n}{\int_M e^{-\frac{|\bar{y}_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}}d\bar{y}_n}$$

$$\cdots$$

$$\frac{\int_M e^{-\frac{|u_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}}du_1}{\int_M e^{-\frac{|\bar{u}_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}}d\bar{u}_1}\cdots\frac{\int_M e^{-\frac{|u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}}du_{n-1}}{\int_M e^{-\frac{|\bar{u}_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}}d\bar{u}_{n-1}}\frac{\int_M e^{-\frac{|u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}}du_n}{\int_M e^{-\frac{|\bar{u}_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}}d\bar{u}_n}$$

$$\frac{\int_M e^{-\frac{|v_1-u_1|^2}{2\Delta s_k\Delta t_1}}dv_1}{\int_M e^{-\frac{|\bar{v}_1-u_1|^2}{2\Delta s_k\Delta t_1}}d\bar{v}_1}\cdots\frac{\int_M e^{-\frac{|v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}}dv_{n-1}}{\int_M e^{-\frac{|\bar{v}_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}}d\bar{v}_{n-1}}\frac{\int_M e^{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}p(v_n)dv_n}{\int_M e^{-\frac{|\bar{v}_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}}d\bar{v}_n}$$

where $\Delta t_i=t_i-t_{i-1}$, $\Delta s_j=s_j-s_{j-1}$. Denote this integral by $I(\mathcal{P}_1,\mathcal{P}_2,f)$. Let the constant K be such that $\|p\|\leqslant K$. Then all the integrals in $I(\mathcal{P}_1,\mathcal{P}_2,f)$ and the integral $I(\mathcal{P}_1,\mathcal{P}_2,f)$ are bounded by the same constant. Without loss of generality we can assume that $\lambda_M(M)=1$.

Lemma 2. *$I(\mathcal{P}_1,\mathcal{P}_2,f)$ converges to $e^{-\frac{st}{2}\Delta_M}f$ as the meshes $|\mathcal{P}_1|$ and $|\mathcal{P}_2|$ tend to zero.*

Proof. First we prove the lemma for such functions p that $e^{-\frac{t}{2}\Delta_M}p$ can be presented by the converging exponential series for all $t\geqslant 0$. According to the book [16], the set of all such functions is dense in $C(M,\mathbb{R})$. If the lemma is proved for such functions then, by the Banach-Steinhaus theorem, the statement of the lemma holds for all $p\in C(M,\mathbb{R})$.

Consider first the bottom line of the integral $I(\mathcal{P}_1,\mathcal{P}_2,p)$ for $u_1,\ldots,u_n$, satisfying the assumption $|u_i-u_{i-1}|<(|\mathcal{P}_2|\Delta t_i)^\alpha$, $\alpha>0$. We will apply the asymptotic (11) from Proposition 2. We chose the number N so that $(|\mathcal{P}_2|)^{\alpha N}<\Delta s_k|\mathcal{P}_2|^\alpha$. Show for example, the applying of the asymptotic of the most right integral.

$$\frac{\int_M e^{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}} p(v_n)dv_n}{\int_M e^{-\frac{|\bar v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}} d\bar v_n} = p(v_{n-1}+(u_n-u_{n-1})_M^{v_{n-1}})$$
$$-\frac{\Delta s_k\Delta t_n}{2}\Delta_M p(v_{n-1}+(u_n-u_{n-1})_M^{v_{n-1}})$$
$$+\mathcal{R}(v_{n-1},u_n-u_{n-1})\, p(v_{n-1}+(u_n-u_{n-1})_M^{v_{n-1}})$$
$$+\Delta s_k\Delta t_n R_2(\Delta s_k\Delta t_n,|\mathcal{P}_2|,v_{n-1},u_n-u_{n-1}),$$

Further we will integrate only the first three terms in this expression, since by Proposition 2 $|R_2(\Delta s_k\Delta t_n,|\mathcal{P}_2|,v_{n-1},u_n-u_{n-1})| < K(\mathcal{P}_2\Delta t_n)^\alpha$ where K is a constant. For each term of the form $(\Delta s_k)^l\Delta t_{i_1}\ldots\Delta t_{i_l}$, $i_1>\cdots>i_l$ we define the coefficient depending on u_1, …, u_n, which appears after applying the asymptotic (11). For convenience of presenting these coefficients we introduce the following notations. Let $y\in M$.

$$y_1 = y,$$
$$y_2 = y_1+(u_{i+1}-u_i)_M^{y_1},$$
$$y_3 = y_2+(u_i-u_{i-1})_M^{y_2},$$
$$\cdots$$
$$y_{n-i+1} = y_{n-i}+(u_n-u_{n-1})_M^{y_{n-i}}.$$

Further, if g is a function on M, then define $g_{(u_nu_{i+1})_M}(y)$ by formula

$$g_{(u_nu_{i+1})_M}(y) = g(y_{n-i+1}).$$

First we calculate the terms of the order not exceeding the order one.

$$p(u_n)-\frac{\Delta s_k\Delta t_n}{2}\Delta_M p(u_n)-\sum_{i=1}^{n-1}\frac{\Delta s_k\Delta t_{n-i}}{2}\Delta_M p_{(u_nu_{n-i+1})_M}(u_{n-i}).$$

Then calculate the coefficient of the term of the form $(\Delta s_k)^l\Delta t_{i_1}\ldots\Delta t_{i_l}$, $i_1>\cdots>i_l$:

$$\frac{1}{2^l}\Delta_M\Big(\ldots\Big(\Delta_M p_{(u_nu_{i_1+1})_M}\Big)_{(u_{i_1}u_{i_2+1})_M}\ldots\Big)_{(u_{i_{l-1}}u_{i_l+1})_M}(u_{i_l}) \qquad (19)$$

Now calculate the terms appearing due to the presence of the factor $\mathcal{R}(v_{n-1},u_n-u_{n-1})$. Note that all the terms containing either summands of the form (12) or sums of products of expressions of the form (12), after applying the integrals of the last line contain an expression of the form $(u_{i+1}-u_i)_\perp^{u_i}$. Note that $(u_{i+1}-$

$u_i)_\perp^{u_i} = 0$, and hence, such terms are equal to zero. Thus, we present $R(v_{n-1}, u_n - u_{n-1})$ in the from

$$\mathcal{R}(v_{n-1}, u_n - u_{n-1}) = (\Delta_M \imath(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}), (u_n - u_{n-1})_\perp^{v_{n-1}})^2 + \mathcal{R}_3(v_{n-1}, u_n - u_{n-1})$$

where

$$\mathcal{R}_3(v_{n-1}, u_n - u_{n-1}) = \sum_{n=3}^{N} \sum k(n) \prod_{l_1+\dots+l_s=n} D_M^{2,l_i}(\imath(\cdot), (u_n - u_{n-1})_\perp^{v_{n-1}})^{l_i} \big(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}\big).$$

Applying the operator Δ_M to $\mathcal{R}_3(v_{n-1}, u_n - u_{n-1})$ always contains an expression $(u_n - u_{n-1})_\perp^{v_{n-1}}$ of the order $|\mathcal{P}_2|^\alpha$. Besides that the operator Δ_M appears with the factor $\Delta s_k \Delta t_{n-1}$. This summand will be included in the rest term which is estimated from above by $K\Delta s_k \Delta t_{n-1}|\mathcal{P}_2|^\alpha$ where K is a constant. Applying the asymptotic to the second integral from the end gives the term $\mathcal{R}_3(v_{n-2}, u_{n-1} - u_{n-2})$. The next applying of the operator Δ_M to this term gives another term which can be estimated from above by $K\Delta s_k \Delta t_{n-2}|\mathcal{P}_2|^\alpha$, and so on.

Show that

$$|\Delta_M(\Delta_M \imath(v_{n-1} + (u_n - u_{n-1})_M^{v_{n-1}}), (u_n - u_{n-1})_\perp^{v_{n-1}})^2| < K|u_n - u_{n-1}|$$

where the first from the left operator Δ_M is applied to the function of the variable v_{n-1}, K is a constant. First consider the case $v_{n-1} \neq u_{n-1}$. First find $\Delta_M(u_n - u_{n-1})_\perp^{v_{n-1}}$ differentiating in v_{n-1}. Find geodesics $\gamma_i(t)$ starting from the point v_{n-1} and such that $\gamma_i'(0)$ build an orthonormal basis in the space tangent to M at the point v_{n-1}. To show that

$$\Big|\frac{d^2}{dt^2}(\Delta_M \imath\big(\gamma_i(t) + (u_n - u_{n-1})_M^{\gamma_i(t)}), (u_n - u_{n-1})_\perp^{\gamma_i(t)}\big)^2\big|_{t=0}\Big| < K|u_n - u_{n-1}|,$$

it suffices to prove that

$$|\frac{d}{dt}(u_n - u_{n-1})_\perp^{\gamma_i(t)}\big|_{t=0}| < K|u_n - u_{n-1}|. \tag{20}$$

Indeed, it is easy to notice that in the case $v_{n-1} \neq u_{n-1}$, $(u_n - u_{n-1})_\perp^{v_{n-1}}$ has the form

$$(u_n - u_{n-1})_\perp^{\gamma_i(t)} = k(t)|u_n - u_{n-1}|n_M(\gamma_i(t))$$

where $k(t)$ is the scalar function which is not bigger than 1 and has a bounded derivative, $n_M(x)$ is a unit vector in the space normal to M at the point x. From this it can be observed that the derivative at zero is proportional to $|u_n - u_{n-1}|$, and hence (20) holds.

Further we prove that

$$|\Delta_M(\Delta_M\imath(v_{n-1}+(u_n-u_{n-1})_M^{v_{n-1}}),(u_n-u_{n-1})_\perp^{v_{n-1}})^2|_{v_{n-1}=u_{n-1}}|<K|u_n-u_{n-1}|.$$

Consider tangent spaces $T_{u_{n-1}}$ and T_{u_n} at the points u_{n-1} and u_n respectively. In each of these tangent spaces we consider the space parallel to the intersection of $T_{u_{n-1}}$ and T_{u_n}. These subspaces have codimension 1 in $T_{u_{n-1}}$ and T_{u_n} respectively. Denote these subspaces by $T'_{u_{n-1}}$ and T'_{u_n} respectively. At the points u_{n-1} and u_n we choose geodesics γ_i^{n-1} and γ_i^n such that the vectors of their derivatives at zero build orthonormal bases in the spaces $T_{u_{n-1}}$ and T_{u_n} respectively, so that the first $d-1$ vectors of the bases are in the spaces $T'_{u_{n-1}}$ and T'_{u_n} respectively (the tangent spaces have the dimension d). As before we have to show (20). Consider the two dimensional curve which is the intersection of the manifold M with the plane spanned on the vectors $(\gamma_i^{n-1})'(0)$ and u_n-u_{n-1}. We see that $\frac{d}{dt}(u_n-u_{n-1})_\perp^{\gamma_i^{n-1}(t)}|_{t=0}=0$ for $i=1,\dots,d-1$. The vector $e_d=(\gamma_d^{n-1})'(0)$ is orthogonal to $T'_{u_{n-1}}$. Let P be the intersection point of the line $u_{n-1}+te_d$ with $T_{u_{n-1}}\cap T_{u_n}$. Set the line from the point P to the point u_n. We got the triangle $\triangle(u_{n-1},P,u_n)$. All the sines of acute angles are proportional to u_n-u_{n-1}. From the geometrical consideration it follows that $|(u_n-u_{n-1})_\perp^{\gamma_i(t)}|<K\,t\,|u_n-u_{n-1}|$, K is a constant, which proves (20).

Thus, under the assumption that $|u_{i+1}-u_i|<(\Delta t_i|\mathcal{P}_2|)^\alpha$ we obtained the general term of the series which appears after applying the last line in the integral $I(p,\mathcal{P}_1,\mathcal{P}_2)$ (the coefficient (19) multiplied by $(\Delta s_k)^l\Delta t_{i_1}\dots\Delta t_{i_l}$, $i_1>\dots>i_l$), and proved that the remaining term does not exceed

$$K\,t\,\Delta s_k\,|\mathcal{P}_2|^\alpha|\mathcal{P}_1|^\alpha \tag{21}$$

where K is a constant.

Starting from the first line in the integral $I(p,\mathcal{P}_1,\mathcal{P}_2)$, we will replace integration over the whole manifold by integration over sufficiently small neighborhoods in the manifold. Then, we will pass to the iterative limit: first to the limit as $|\mathcal{P}_1|\to 0$, and then to the limit as $|\mathcal{P}_2|\to 0$. Describe the choice of the neighborhoods mentioned above. Chose $0<\alpha<\tilde{\alpha}<\frac{1}{2}$, $\Delta\alpha=\tilde{\alpha}-\alpha$. In the integral of the form $\frac{\int_M e^{-\frac{|x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}h(x_i)dx_i}{\int_M e^{-\frac{|\bar{x}_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}d\bar{x}_i}$ which stays at the ith position in the first line we replace the integration over the whole manifold with the integration over the neighborhood $U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}})$ of the point x_{i-1} of radius $|\mathcal{P}_2|^{\tilde{\alpha}}(\Delta t_i)^{\tilde{\alpha}}$. Here h denotes the general form a function to which we apply the above integral operator (in our situation this function is a sequence of integrals applied to the function p).

We have

$$\frac{1}{(2\pi\Delta s_1\Delta t_i)^{\frac{d}{2}}}\int\limits_{M\setminus U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha})} e^{-\frac{|x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}h(x_i)\,dx_i < \frac{C}{(2\pi\Delta s_1\Delta t_i)^{\frac{d}{2}}}\,e^{-\frac{1}{(\Delta s_1\Delta t_i)^{1-2\tilde\alpha}}}$$

where the constant C is such that $|p(y)| < C$ for all $y \in M$ (and hence, the functions under the integral sign of each integral in $I(p, \mathcal{P}_1, \mathcal{P}_2)$ are bounded by the same constant); as we have mentioned above $\lambda(M) = 1$ without loss of generality. By Corollary 1 for sufficiently small mesh of the partition $\mathcal{P}_1$, we get

$$\frac{\int\limits_{M\setminus U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha})} e^{-\frac{|x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}h(x_i)\,dx_i}{\int_M e^{-\frac{|\bar x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}d\bar x_i} < \frac{C}{(2\pi\Delta s_1\Delta t_i)^{\frac{d}{2}}}\,e^{-\frac{1}{(\Delta s_1\Delta t_i)^{1-2\tilde\alpha}}}$$

Thus, each ith integral of the first line we replace with the integral of the form

$$\frac{\int\limits_{U_{x_{i-1}}(|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha})} e^{-\frac{|x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}h(x_i)\,dx_i}{\int_M e^{-\frac{|\bar x_i-x_{i-1}|^2}{2\Delta s_1\Delta t_i}}d\bar x_i},$$

where the general form of the function h is described above. Further, for the ith integral of the second line we have $|x_i - x_{i-1}| < |\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}$. Choose a neighborhood $U_{y_{i-1}}(2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}) \subset M$ of the point y_{i-1} such that for all points y_i of this neighborhood the estimate $|y_i - y_{i-1}| < 2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}$ holds. We have

$$|y_i - y_{i-1} - (x_i - x_{i-1})| > |y_i - y_{i-1}| - |x_i - x_{i-1}| > (2^{\Delta\alpha} - 1)|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha}\ .$$

Taking into account this inequality and Proposition 2, similar to the above estimate we get

$$\frac{\int_{M\setminus U_{y_{i-1}}(2^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha})} e^{-\frac{|y_i-y_{i-1}-x_i+x_{i-1}|^2}{2\Delta s_2\Delta t_i}}h(y_i)\,dy_i}{\int_M e^{-\frac{|\bar y_i-y_{i-1}-x_i+x_{i-1}|^2}{2\Delta s_2\Delta t_i}}d\bar y_i} < \frac{C}{(2\pi\Delta s_2\Delta t_i)^{\frac{d}{2}}}\,e^{-\frac{(2^{\Delta\alpha}-1)^{\tilde\alpha}}{(\Delta s_2\Delta t_i)^{1-2\tilde\alpha}}}\ .$$

We continue choosing neighborhoods in this way, finally we will consider the ith integral of the last line for the points u_i and u_{i-1} of the form $|u_i - u_{i-1}| < (k-1)^{\Delta\alpha}|\mathcal{P}_2|^{\tilde\alpha}(\Delta t_i)^{\tilde\alpha} < |\mathcal{P}_2|^{\alpha}(\Delta t_i)^{\alpha}$ where the last inequality holds for a sufficiently small mesh of the partition $|\mathcal{P}_1|$, so that $(k|\mathcal{P}_2|\Delta t_i)^{\Delta\alpha} < 1$.

Further staring from the last integral in the last line, we apply the asymptotic (11) from Proposition 2, as we described it above.

Now consider the second from the bottom line of the integral operators. As before, for a sufficiently small mesh of the partition $\mathcal{P}_1$ we can take $|z_i - z_{i-1}| < (\Delta t_i\, |\mathcal{P}_2|)^\alpha)$ for all i. We will act with the sequence of the integral operators on each term obtained after decomposition of the last line, except the rest term. Here we apply the asymptotic decomposition of Proposition 2. Applying the first $k-1$ lines to the rest term of the last line gives a new rest term which can be also estimated by (21). For this reason we exclude this term from our consideration. Further, we act with the third from the bottom line of integral operators on the terms obtained from the two bottom lines, and so on. At the end we apply the integral operators of the first line. Thus, we get an asymptotic decomposition of the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$. Calculate the general term of this decomposition (up to the sign):

$$\frac{(\Delta s_k)^{l_0}\Delta t_{i_1}\cdots\Delta t_{i_{l_0}}(\Delta s_{k_1})^{l_1}\Delta t_{j_1}\cdots\Delta t_{j_{l_1}}\cdots(\Delta s_{k_s})^{l_s}\Delta t_{m_1}\cdots\Delta t_{m_{l_s}}}{2^{l_0+l_1+\cdots+l_s}}$$
$$\times\left(\Delta_M^{(y_1^s)}\right)^{l_{s-q}}\cdots\left(\Delta_M^{(y_q^s)}\right)^{l_s}\Big(\cdots\left(\Delta_M^{(y_1^1)}\right)^{l_{r+1}}\cdots\left(\Delta_M^{(y_p^1)}\right)^{l_p}\Big(\left(\Delta_M^{(y_1^0)}\right)^{l_1}\cdots\left(\Delta_M^{(y_r^0)}\right)^{l_r}$$
$$\left(\Delta_M^{l_0} p_{\sum_i (y_i^0 - x)_M}(x)\right)_{\sum_i (y_i^1 - x)_M}\Big|_{y^0 = x}\Big)_{\sum_i (y_i^2 - x)_M}\Big|_{y^1 = x}\cdots\Big|_{y^{s-1}=x}\Big)\Big|_{y^s = x} \quad (22)$$

Here $l_i \geqslant 0$, $i = 1, \ldots, s$, $k > k_1 > \cdots > k_s$, the index (y_i^j) on the top means that the operator Δ_M^p is applied to the function of the variable y_i^j, $y^i = x$ means that $y_j^i = x$ for all j. For an arbitrary function φ defined on M, $\varphi_{\sum_i (w_i)_M}$ means $\varphi_{\sum_i (w_i)_M}(y) = \varphi(y + (w_1)_M^y + (w_2)_M^{y+(w_1)_M^y} + \cdots)$, where w_i are sufficiently small so that $\varphi_{\sum_i (w_i)_M}$ is defined correctly. Calculation of (22) in local coordinates fixed in a neighborhood of the point x gives

$$\frac{(\Delta s_k)^{l_0}\Delta t_{i_1}\cdots\Delta t_{i_{l_0}}(\Delta s_{k_1})^{l_1}\Delta t_{j_1}\cdots\Delta t_{j_{l_1}}\cdots(\Delta s_{k_s})^{l_s}\Delta t_{m_1}\cdots\Delta t_{m_{l_s}}}{2^{l_0+l_1+\cdots+l_s}}\,\Delta_M^{l_0+l_1+\cdots+l_s}(x)$$

Compare the decomposition of $I(\mathcal{P}_1, \mathcal{P}_2, \cdot)$ with the decomposition of

$$e^{-\frac{st}{2}\Delta_M} = e^{-\frac{\Delta s_1 \Delta t_1}{2}\Delta_M}\cdots e^{-\frac{\Delta s_k \Delta t_n}{2}\Delta_M},$$

if we apply the asymptotic $e^{-\frac{\Delta s_i \Delta t_j}{2}\Delta_M} = 1 - \frac{\Delta s_i \Delta t_j}{2}\Delta_M + O((\Delta s_i \Delta t_j)^{\frac{3}{2}})$ subsequently to each exponent started from the right. Up to the terms tending to zero as $|\mathcal{P}_1| \to 0$, $|\mathcal{P}_2| \to 0$, the decompositions coincide.

Due to the symmetry of the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$ we can first take the limit in n with the fixed k, and then the limit in k. The both iterative limits coinside and equal to $e^{-\frac{st}{2}\Delta_M}$. Hence, as $|\mathcal{P}_1| \to 0$, $|\mathcal{P}_2| \to 0$, there exist a double limit and equal to $e^{-\frac{st}{2}\Delta_M}$. □

If the function f is such that $f(\omega)=p(\omega(\tilde t,\tilde s))$, where $\tilde t<t$, $\tilde s<s$,

$$\int\limits_{C([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)=I(p,\mathcal{P}_1,\mathcal{P}_2)+\delta(x,\tilde t-t_n,\tilde s-s_n)$$

where s_n and t_n are partition points of $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\tilde t\in(t_n,t_{n+1})$, $\tilde s\in(s_n,s_{n+1})$. If σ is an arbitrary sufficiently small number, then for sufficiently small meshes of $\mathcal{P}_1$ and $\mathcal{P}_2$ the inequality $|\delta(x,(\tilde t-t_n,\tilde s-s_n))|<\sigma$ holds. For a function f depending on ω in several points, say in points $\xi_i\in[0,s]$ $\tau_j\in[0,t]$, the form of the integral $I(\mathcal{P}_1,\mathcal{P}_2,p)$ is the same and the convergence can be proved analogously. The integral converges to a product of operators of the form $e^{-\frac{\Delta\xi_i\Delta\tau_j}{2}\Delta_M}$, each operator acts on the corresponding variable of the function p, defined as $f(\omega)=p(\omega_{11}(\xi_1,\tau_1),\ldots,\omega_{kl}(\xi_l,\tau_l))$, where ω_{ij} is defined on $[0,\xi_i-\xi_{i-1}]\times[0,\tau_j-\tau_{j-1}]$ by the formula $\omega_{ij}(s,t)=\omega(\xi_{i-1}+s,\tau_{j-1}+t)$. The theorem is proved. □

Corollary 2. *Let M be a compact Lie group. Then $\mathbf{W}^x_M$ being considered as a process with values in $C([0,1],M)$, coincides with a $C([0,1],M)$-valued Brownian motion defined in [7].*

Proof. The proof follows from Theorem 1 and Theorem 2.15 from the paper [11]. Indeed, Theorem 2.15 from the paper [11] together with Theorem 1 imply the coincidence of the finite dimensional distributions of the process $\mathbf{W}^x_M$ and the $C([0,1],M)$-valued Brownian motion defined in [7]. Indeed, according to Theorem 2.15 from the paper [11] for the process defined in [7] the following statement holds: for each fixed s it is a multiple Brownian motion on M with the parameter s. The coincidence of the finite distributions follows now from a formula in the book [9] on the page 204. Thus, the distributions of the processes coincide on the algebra of cylindric subsets of the space $C([0,1],(C([0,1],M))$, where in $C([0,1],M)$ we also fix the algebra of cylindric subsets. This implies the σ-additivity of the measure $\mathbb{W}^x_M$ since the distribution of the Brownian motion defined in [7] is a σ-additive measure. Thus, $\mathbb{W}^x_M$ is defined on the σ-algebra of all Borelean subsets of the space $C([0,1],C([0,1],M))$, σ-additive and coincides with the distribution of a Brownian motion from the paper [7]. □

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